LECTURE 4: JUNG'S THEOREM AND MINKOWSKI SUMS

In this lecture we start with proving an application of Helly's theorems. Let us recall Jung's theorem.

Theorem 1 (Jung)**.** *A set in $\mathbb{R}^n$ having diameter d can be covered by a closed Euclidean ball of radius $d \cdot \sqrt{\frac{n}{2(n+1)}}$.*

We remark that the quantity in the theorem is the circumradius of the regular n-dimensional simplex of edge length d.

Proof. Let the diameter of $S \subset \mathbb{R}^n$ be d, and for every $p \in S$, let G_p denote the set of points x in $\mathbb{R}^n$ with the property that the closed ball of radius $d \cdot \sqrt{\frac{n}{2(n+1)}}$ and center x covers p. Note that G_p is the closed ball of radius $d \cdot \sqrt{\frac{n}{2(n+1)}}$ centered at p (bot conditions are equivalent to saying that $||x-p|| \leq d \cdot \sqrt{\frac{n}{2(n+1)}}$), and thus, it is compact and convex. Hence, if we can verify that $\bigcap_{i=1}^{k} G_{p_i} \neq \emptyset$ for any $p_1, p_2, \ldots, p_k \in S$ and $k \leq n+1$, then from Helly's theorem (infinite version) it follows that $\bigcap_{p \in S} G_p \neq \emptyset$, which readily yields our theorem.

Let $p_1, p_2, \ldots, p_k \in S$ with $k \leq n+1$, and let q be the center of the smallest closed ball containing the points $p_1, p_2, \ldots, p_k$. We show that the radius of G is at most $r \leq d\sqrt{\frac{n}{2(n+1)}}$. Observe that $q \in \operatorname{conv}\{p_1, p_2, \ldots, p_{n+1}\}$, as otherwise there is a smaller ball that contains the points. Without loss of generality, we may assume that $q = o$. In addition, since we have only finitely many points, G is the smallest ball that contains those p_is that are contained in its boundary, and thus, we may assume that $||p_i|| = \sqrt{\langle p_i, p_i \rangle} = r$ for all values f i. As the diameter of S is d, we have $||p_i - p_j|| = \operatorname{dist}(p_i, p_j) \leq d$ for all i and j. Thus,

$$d^2 \geq \langle p_i - p_j, p_i - p_j \rangle = ||p_i||^2 + ||p_j||^2 - 2\langle p_i, p_j \rangle = 2r^2 - 2\langle p_i, p_j \rangle.$$

As $o = q = \sum_{i=1}^{k} \alpha_i p_i$, where $\alpha_i \geq 0$ and $\sum_{i=1}^{k} \alpha_i = 1$, we obtain that for all values of i,

$$1 - \alpha_i = \sum_{j \neq i} \alpha_j \geq \sum_{j \neq i} \alpha_j \frac{\langle p_i - p_j, p_i - p_j \rangle}{d^2} = \sum_{j \neq i} \alpha_j \frac{2r^2 - 2\langle p_i, p_j \rangle}{d^2},$$

where from the equality $\langle p_i, p_i \rangle = r^2$ it follows that

$$1 - \alpha_i \geq 2 \sum_{j \neq i} \alpha_j \frac{r^2}{d^2} - 2 \sum_{j \neq i} \frac{\langle p_i, p_j \rangle \alpha_j}{d^2} + 2\alpha_i \frac{r^2 - \langle p_i, p_i \rangle}{d^2} =$$

$$= 2\frac{r^2}{d^2} \sum_{j=1}^{k} \alpha_j - \frac{2}{d^2} \left\langle p_i, \sum_{j=1}^{k} \alpha_j p_j \right\rangle = \frac{2r^2}{d^2}.$$

Summing up for all is, we obtain that

$$k - 1 = \sum_{i=1}^{k} (1 - \alpha_i) \geq k \frac{2r^2}{d^2},$$

from which, as $k \leq n + 1$, the inequality

$$r^2 \leq \frac{k-1}{2k} d^2 \leq \frac{n}{2n+2} d^2$$

follows. □

To continue, recall the definition of the Minkowski sum of two sets from the first lecture.

Definition 1 (Lecture 1, Definition 1)**.** *Let V_1 and V_2 be two point sets, and let $\lambda \in \mathbb{R}$. Then*

$$V_1 + V_2 = \{v_1 + v_2 : v_1 \in V_1, v_2 \in V_2\}$$

is the Minkowski sum of the two sets and

$$\lambda V_1 = \{\lambda v_1 : v_1 \in V_1\}$$

is the multiple of V_1 by λ.

Remark 1. *To 'draw' the Minkowski sum of two sets we should think it over that by definition, $V_1 + V_2 = \bigcup_{v_1 \in V_1} (v_1 + V_2)$, implying that the sum of the two sets can be obtained as the region 'swept' by the translates of one of the sets where the translation vectors run over the other set.*

Proposition 1. *If $K, L \subset \mathbb{R}^n$ are convex, then $K + L$ is convex.*

Proof. We need to show that the segment connecting any two points of $K+L$ belongs to $K+L$. In other words, we need to show that if $p_k, q_k \in K$ and $p_L, q_L \in L$, then $[p_K+p_L, q_K+q_L] \subseteq K+L$. Let $t \in [0,1]$ be arbitrary. Then $t(p_K+q_K)+(1-t)(p_L+q_L) = (tp_K+(1-t)q_K) + (tp_L+(1-t)q_L)$, where by the convexity of K and L, we have $tp_K+(1-t)q_K \in K$ and $tp_L+(1-t)q_L \in L$. Thus, $t(p_K+q_K)+(1-t)(p_L+q_L) \in K+L$, from which the statement follows. □

Definition 2. *Let $A \subset \mathbb{R}^n$ be an arbitrary bounded set. Then the function*

$$h_A : \mathbb{R}^n \to \mathbb{R}, \quad h_A(x) = \sup\{\langle x, y\rangle : y \in A\}$$

is called the support function *of A.*

We finish with stating a theorem about support function, which we are going to prove next week.

Theorem 2. *Let $A \subset \mathbb{R}^n$ be an arbitrary bounded set containing o. Then the support function h_A of A is:*

(i) *convex, that is, $h(tx+(1-t)y) \leq th(x)+(1-t)h(y)$ for every $x, y \in \mathbb{R}^n$ and $t \in [0,1]$;*
(ii) *h nonnegative, and for any $\lambda \geq 0$ and $x \in \mathbb{R}^n$, we have $h(\lambda x) = \lambda h(x)$.*

Furthermore, for any function h satisfying the above properties there is a unique compact, convex set $A \subset \mathbb{R}^n$, containing o, whose support function is h.