LECTURE 6: SEPARATION

Remark 1. *Let $L_1, L_2 \subseteq \mathbb{R}^n$ be linear subspaces with* $\dim(L_1) = k$ *and* $\dim(L_2) = n - k$ *for some $0 \leq k \leq n$, and let $L_1 \cap L_2 = \{o\}$. Then the union of a basis of L_1 and a basis of L_2 is a basis of $\mathbb{R}^n$, and hence, for any point $p \in \mathbb{R}^n$ there are unique points $p_1 \in L_1$, $p_2 \in L_2$ satisfying $p = p_1 + p_2$.*

Definition 1. *Let $L_1, L_2 \subseteq \mathbb{R}^n$ be linear subspaces with* $\dim(L_1) = k$ *and* $\dim(L_2) = n - k$ *for some $0 \leq k \leq n$, and let $L_1 \cap L_2 = \{o\}$. For any $x \in \mathbb{R}^n$ let $x_1 \in L_1$, $x_2 \in L_2$ denote those unique points that satisfy $x = x_1 + x_2$. Then the linear transformation $\pi : \mathbb{R}^n \to L_2$, $\pi(x) = x_2$ is called* projection onto L_2 parallel to L_1. *If L_1 is the orthogonal complement of L_2, then we say that π is the* orthogonal projection onto L_2.

From the definition it is clear that if $\dim(L_1) = k$ and L is an affine subspace of dimension m in L_2, then $\pi^{-1}(L)$ is an $(m+k)$-dimensional affine subspace in $\mathbb{R}^n$.

Remark 2. *If the conditions of the previous remark are satisfied for the linear subspaces $L_1, L_2 \subseteq \mathbb{R}^n$ then for any $p_1, p_2 \in \mathbb{R}^n$, the intersection of $p_1 + L_1$ and $p_2 + L_2$ is a singleton. Indeed, by the previous remark, p_1 can be decomposed to the sum of a vector from L_1 and a vector from L_2,and hence, as $x + L_1 = L_1$ if $x \in L_1$, we may assume that $p_1 \in L_2$. Similarly, we may assume that $p_2 \in L_1$. Thus, if $x \in \mathbb{R}^n$ is contained in both subspaces, then, writing it in the form $x = x_1 + x_2$, $x_1 \in L_1$, $x_2 \in L_2$, the previous remark implies that $x_1 = p_2$ and $x_2 = p_1$; on the other hand $p_1 + p_2$ is an element of both subspaces. Based on this observation, projection can be defined not only for linear subspaces, but also for affine subspaces.*

Proposition 1. *Let $L_1, L_2 \subseteq \mathbb{R}^n$ be linear subspaces with* $\dim(L_1) = k$ *and* $\dim(L_2) = n - k$ *for some $0 \leq k \leq n$, and let $L_1 \cap L_2 = \{o\}$. Let π be the projection onto L_2 parallel to L_1. Then for any open/compact/convex set $X \subset \mathbb{R}^n$, $\pi(X)$ is open/compact/convex, respectively, and for any open/closed/convex set $Y \subseteq L_2$, the set $\pi^{-1}(Y)$ is open/closed/convex, respectively.*

Proof. For any point $x \in \mathbb{R}^n$ the projection of a neighborhood of x is a neighborhood of $\pi(x)$ in L_2, and hence, if $X \subseteq \mathbb{R}^n$ open, then $\pi(X)$

is also open. Similarly, the projection of a closed segment is the closed segment connecting the projections of the endpoints, which yields that if X is convex, then so is $\pi(X)$. The statement for the projection of a compact set follows from the observation that π is a continuous function, and thus, the image of a compact set is compact. Similarly, it also follows that the preimage of an open/closed set is open/closed, respectively. Now, if $Y \subset L_2$ is convex, then for any $p, q \in Y$ choose some points $p', q' \in \mathbb{R}^n$ $\pi(p') = p$, $\pi(q') = q$. As $\pi([p', q']) = [p, q] \subseteq Y$ by the convexity of Y, we clearly have $[p', q'] \subseteq \pi^{-1}([p, q])$, implying that $\pi^{-1}(Y)$ is convex. □

Definition 2. *Let $A, B \subseteq \mathbb{R}^n$. Let H be a hyperplane, and let H^+ and H^- be the two closed half spaces bounded by H. We say that H* separates *A and B if $A \subseteq H^+$ and $B \subseteq H^-$, or $B \subseteq H^+$ and $A \subseteq H^-$. If H separates A and B, and $A \cap H = B \cap H = \emptyset$, then we say that H* strictly separates *A and B. If $A \subseteq H$, and $B \subseteq H^+$ or $B \subseteq H^-$, then we say that H* isolates *A from B. If, in addition, $B \cap H = \emptyset$, then we say that H* strictly isolates *A from B.*

Theorem 1 (Isolation theorem)**.** *Let $K \subseteq \mathbb{R}^n$ be an open, convex set, and let $o \notin K$. Then there is a hyperplane H that isolates o from K.*

We remark that if a hyperplane H isolates of from K, then by the openness of K it also strictly isolates o from K.

Proof. The statement is trivial if $n = 1$. First, we prove it for $n = 2$. Let $\mathbb{S}^1$ be the set of unit vectors in $\mathbb{R}^2$, i.e. let it be the boundary of the circular disk centered at o and with unit radius. Let $p : \mathbb{R}^2 \setminus \{o\} \to \mathbb{S}^1$ be the central projection onto $\mathbb{S}^1$, i.e. let $p(v) = \frac{v}{||v||}$. Since K is convex, therefore it is connected, and thus, $p(K)$ is also connected. It is also clear that since K is open, the set $p(K)$ is also open. Thus, $p(K)$ is an open circular arc in $\mathbb{S}^1$. If $p(K)$ contains two opposite points $u, -u$, then there would be positive real numbers $\lambda_1, \lambda_2 > 0$ with $\lambda_1 u, -\lambda_2 u \in K$. But this would imply by the convexity of K that $o \in K$, which contradicts our assumptions. Hence, $p(K)$ does not contain opposite points, which yields that the length of $p(K)$ is at most π, or in other words, there are opposite points $u, -u \in \mathbb{S}^1$ such that neither one belongs to $p(K)$. This yields that there is a line through o disjoint from K.

If $n > 2$, we prove the statement by induction on n. Assume that the statement holds in $\mathbb{R}^k$ for every $1 \leq k < n$.

Consider a plane P through o. Since $P \cap K$ is an open, convex set, we may apply the case $n = 2$ of the statement and obtain a line $L \subset S$ through o disjoint from K. Let $H = L^\perp$ be the orthogonal complement of L. Let π be the orthogonal projection onto H (parallel to L). Then by Proposition 1, $\pi(K)$ is an open, convex set in H, and thus, by the induction hypothesis, there is some $(n-2)$-dimensional linear subspace $G \subset H$ $(n-2)$ disjoint from $\pi(K)$. But then $\pi^{-1}(G)$ is a hyperplane H' in $\mathbb{R}^n$, which contains o and is disjoint from $\pi^{-1}(\pi(K))$, and in particular from K. Thus, by the convexity of K, H' isolates o from K. □

The question arises whether a point can be isolated from convex sets in general. To be able to answer this question, we first prove some lemmas.

Lemma 1. *If $K \subseteq \mathbb{R}^n$ is convex and $\operatorname{int}(K) \neq \emptyset$, then $K \subseteq \operatorname{cl}(\operatorname{int}(K))$.*

This statement is clearly false if $\operatorname{int}(K) = \emptyset$.

Proof. Let $p \in K$ and $q \in \operatorname{int}(K)$ be arbitrary, where, without loss of generalty, we may assume that $p = o$. As $q \in \operatorname{int} K$, there is some $\varepsilon > 0$ such that the neighborhood of q of radius ε is a subset of K. But then for any point $r \in (o, q)$, the neighborhood of r of radius $\frac{||r||\varepsilon}{||q||}$ is a subset of K, implying that $(o, q) \subset \operatorname{int}(K)$. Thus, $o \in \operatorname{cl}(\operatorname{int}(K))$. □

Lemma 2. *If $K \subset \mathbb{R}^n$ is convex and $\operatorname{int}(K) = \emptyset$, then $\dim(K) < n$, or in other words, there is a hyperplane H with $K \subseteq H$.*

Proof. The proof is based on the observation that if the points $p_1, p_2, \ldots, p_{n+1}$ are affinely independent, then the interior of $\operatorname{conv}\{p_1, \ldots, p_{n+1}$ is not empty: indeed, if, e.g. $\frac{1}{n+1} \sum_{i=1}^{n+1} p_i$ is a boundary point of the convex hull, then by the compactness of the convex hull(2nd lecture, Theorem 4) according to Corollary 4 of the 1st lecture, there is a closed half space containing the convex hull and containing the above point in its boundary, but then by Proposition 1 of the 2nd lecture the bounding hyperplane of this half space contains all of the p_is, which contradicts our assumption that they are affinely independent.

Now, let $p_1, \ldots, p_k$ an affinely independent point system of maximal cardinality in K. Then, by the previous observation, $k \leq n$, implying that there is a hyperplane H containing all of the points. If K has some point $p \notin H$, then it follows from Corollary 1 and Theorem 2 of the 1st lecture that $p_1, \ldots, p_k, p$ are affinely independent, which is in contradiction with the choice of the point system. Thus, $K \subseteq H$. □

Theorem 2 (Isolation theorem 2)**.** *Let $K \subseteq \mathbb{R}^n$ be convex with $o \notin \operatorname{int}(K)$. Then there is a hyperplane H isolating o from K.*

Proof. Assume that $\operatorname{int}(K) \neq \emptyset$. Since $\operatorname{int}(K)$ is convex (Exercise 3 from the first worksheet), by the isolation theorem there is a hyperplane

H that isolates o from $\text{int}(K)$. But then, since closed half spaces are closed sets, H isolates o from $\text{cl}(\text{int}(K))$, and thus, also from K.

Now, let $\text{int}(K) = \emptyset$ and let $G = \text{aff}(K)$. Then the relative interior of K is nonempty in G, and hence, there is an affine subspace G' in G for which $\dim(G') = \dim(G) - 1$, and which isolates o from K in G. But then, choosing any hyperplane H satisfying $G \cap H = G'$, H isolates o from K. □

Theorem 3. *If $K, L \subset \mathbb{R}^n$ are disjoint, convex sets, then K and L can be separated by a hyperplane.*

Proof. Let $M = K - L = K + (-1)L$. Since K and L are disjoint, $o \notin K - L$. But then, by the previous theorem, there is a hyperplane H which isolates o from M. In other words, there is a linear functional $f : \mathbb{R}^n \to \mathbb{R}$ satisfying $f(x) \geq 0$ for any $x \in M$. But then $M = K - L$ implies $0 \leq \inf\{f(x) : x \in M\} = \inf\{f(x) - f(y) : x \in K, y \in L\} = \inf\{f(x) : x \in K\} - \sup\{f(y) : y \in L\}$. Let $\alpha = \inf\{f(x) : x \in K\}$. Then, according to the conditions, for any $x \in K$ we have $f(x) \geq \alpha$, and for any $x \in L$ we have $f(x) \leq \alpha$, and thus, the hyperplane $\{x : f(x) = \alpha\}$ separates K and L. □

Corollary 1. *If $K, L \subset \mathbb{R}^n$ are disjoint, open, convex sets, then K and L can be strictly separated by a hyperplane.*

Problem 1. *Give an example of convex sets $K, L \subset \mathbb{R}^n$ whose interiors are disjoint, but which cannot be separated by a hyperplane.*