LECTURE 7: SEPARATION, FACES OF CONVEX SETS, EXTREMAL AND EXPOSED POINTS, THE KREIN-MILMAN THEOREM

We start with two theorems regarding separation.

Theorem 1. *Let $K, L \subset \mathbb{R}^n$ be convex sets with $\text{int}(K) \neq \emptyset$ and $\text{int}(K) \cap L = \emptyset$. Then K and L can be separated by a hyperplane.*

Proof. We have seen that if K is convex, then $\text{int}(K)$ is convex (Exercise 3 on the first worksheet). But then by last theorem on the last lecture, the sets $\text{int}(K)$ and L can be separated by a hyperplane. Since we learned that if $\text{int}(K) \neq \emptyset$, then $K \subset \text{cl}(\text{int}(K))$, and a hyperplane separating $\text{int}(K)$ and L separates also $\text{cl}(\text{int}(K))$ and L, the assertion follows. □

Theorem 2. *If $K, L \subset \mathbb{R}^n$ are disjoint, convex sets, K is compact and L is closed, then K and L can be strictly separated by a hyperplane.*

Proof. We apply the idea of Theorem 4 in the first lecture. Let $x \in K$ and $y \in L$ be arbitrarily chosen points, and let $r = ||y - x||$. Let L_0 be the set of the points of L whose distance from a point of K is at most r; in other words, let $L_0 = L \cap (K + r\mathbf{B}^n)$, where $\mathbf{B}^n$ is the closed unit ball centered at o. Then the distance between any points of $L \setminus L_0$ and K is greater than r, yielding that $\text{dist}(K, L) = \text{dist}(K, L_0)$, where $\text{dist}(A, B) = \inf\{||a - b|| : a \in A, b \in B\}$. But both K and L_0 are compact sets, and hence, there are points $x \in K$ and $y \in L$ for which $\text{dist}(x, y)$ is minimal. Let H be the hyperplane bisecting the segment $[x, y]$. Then H strictly separates K and L, as otherwise there are points $x' \in K$ and $y' \in L$ for which $||x' - y'|| < ||x - y||$. □

We have already seen (Corollary 4 in the first lecture) that for every boundary point of a convex set there is a hyperplane through the point such that the set is contained in one of the two closed half spaces bounded by the hyperplane. This is the motivation behind the following definitions.

Definition 1. *Let $K \subseteq \mathbb{R}^n$ be a convex set. If H is a closed half space satisfying $K \subseteq H$ and whose boundary intersects the boundary of K, we say that H is a* supporting half space *of K, and the boundary of H is a* supporting hyperplane *of K.*

Definition 2. *Let $K \subseteq \mathbb{R}^n$ be a closed, convex set and let H be a supporting hyperplane of K. Then the set $H \cap K$ is called a* proper face *of K. The empty set is called a not proper face of K. The 0-dimensional faces (consisting of only one point) are called the* exposed points *of K, and their set is denoted by* $\operatorname{ex}(K)$.

Our first observation implies the next remark in a natural way.

Remark 1. *If $K \subseteq \mathbb{R}^n$ is closed and convex, and $p \in \mathbb{R}^n$ is a boundary point of K, then K has a proper face F such that $p \in F$.*

Problem 1. *Construct closed, convex sets which have no exposed points.*

Proposition 1. *If F is a proper face of the closed, convex set $K \subseteq \mathbb{R}^n$, then F is closed and convex.*

Proof. Since every proper face F of K can be written as $F = K \cap H$, where H is a supporting hyperplane of K, and a hyperplane is closed and convex, the assertion follows from the fact that the intersection of closed, convex sets is closed and convex. □

Definition 3. *Let $K \subseteq \mathbb{R}^n$ be closed and convex. If $p \in \operatorname{bd} K$, and for every $q, r \in K$, $p \in [q, r]$ we have $p = q$ or $p = r$, then we say that p is an* extremal point *of K. In other words, the extremal points of K are the points of K that are not relative interior points of a segment in K. The set of the extremal points of K is denoted by* $\operatorname{ext}(K)$.

Proposition 2. *If $K \subseteq \mathbb{R}^n$ is closed and convex, then* $\operatorname{ex}(K) \subseteq \operatorname{ext}(K)$.

Proof. Let p be an exposed point of K. Then there is a linear functional $f : \mathbb{R}^n \to \mathbb{R}$ and a quantity $\alpha \in \mathbb{R}$ such that $K \subset f^{-1}([\alpha, \infty))$, and $K \cap f^{-1}(\alpha) = \{p\}$. Assume that $q, r \in K$ and $p \in [q, r]$. Then there is value $t \in [0, 1]$ with $p = tq+(1-t)r$. By our conditions and the linearity of f, we have $\alpha = f(p) = tf(q) + (1-t)f(r) \geq t\alpha + (1-t)\alpha = \alpha$. But here, inequality occurs if and only if $t = 1$ and $f(q) = \alpha$, or $t = 0$ and $f(r) = \alpha$, or $f(q) = f(r) = \alpha$. But these yield $p = q$, $p = r$, and $p = q = r$, respectively, implying the statement. □

Example. Let $K \subseteq \mathbb{R}^2$ be the union of the unit square $[0, 1]^2$ and the circular region defined by the inequality $(x - 1/2)^2 + y^2 \leq 1/4$.then o and the point $(1, 0)$ are extremal points of K, but not exposed points of L. Thus, there are closed, convex sets K for which $\operatorname{ex}(K)$ and $\operatorname{ext}(K)$ do not coincide.

Our next theorem explores the connection between extremal points and linear functionals.

Theorem 3. *Let $K \subseteq \mathbb{R}^n$ be a closed, convex set, and let $f : \mathbb{R}^n \to \mathbb{R}$ be a linear functional whose minimal or maximal value on K is α. Let $F = K \cap f^{-1}(\alpha)$. Then $p \in F$ is an extremal point of F if and only if it is an extremal point of K. In other words,* $\operatorname{ext}(F) = \operatorname{ext}(K) \cap f^{-1}(\alpha)$.

Before proving Theorem 3, we observe that if $p \in \operatorname{ex}(K)$, then there is a linear functional $f : \mathbb{R}^n \to \mathbb{R}$ which attains its minimum on K only at p. Thus, a consequence of this theorem is the containment $\operatorname{ex}(K) \subseteq \operatorname{ext}(K)$ for every closed, convex set K.

Proof. Assume that $p \in \operatorname{ext}(K)$ and $p \in F$. Then, by the definition of extremal point, for any $q, r \in K$, $p \in [q, r]$ we have $q = p$ or $r = p$. In particular, this holds also for any $q, r \in F$, implying that $p \in \operatorname{ext}(F)$.

Now, let $p \in \operatorname{ext}(F)$, and consider points $q, r \in K$ with $p \in [q, r]$. If $q \neq p$ and $r \neq p$, then for a suitable $t \in (0, 1)$, $p = tq + (1 - t)r$. But from this $\alpha = f(p) = f(tq + (1 - t)r) = tf(q) + (1 - t)f(r)$. As $f(q), f(r) \geq \alpha$, there is equality if and only if $f(q) = f(r) = \alpha$, azaz ha $q, r \in F$. But as $p \in \operatorname{ext}(F)$, thiy yields $q = p$ or $r = p$, which is a contradiction. □

Our next theorem shows an important property of extremal points.

Theorem 4 (Krein, Milman)**.** *Any compact,convex set $K \subset \mathbb{R}^n$ is the convex hull of its extremal points.*

Proof. We prove the statement by induction on the dimension. Assume that $K \subset \mathbb{R}$ is a compact, convex set. Then K is a closed segment, whose extremal points are its endpoints, and the segment is the convex hull of its endpoints. Thus, the assertion holds for $n = 1$.

Assume that the statement is true for any at most $(n-1)$-dimensional compact, convex set, and let K be K an n-dimensional compact, convex set. Let $p \in K$ be arbitrary, and let L bi an arbitrary line through p. According to our conditions, $L \cap K$ is a closed finite segment. Let the endpoints of this segment be q and r, where these points may not be distinct from each other or p. Then, by Remark 1, there are faces F_q and F_r of K such that $q \in F_q$ and $r \in F_r$. But as F_q and F_r are convex subsets of the boundary of K, they have no interior points, and thus, by Lemma 2 of the fourth lecture, they are at most $(n-1)$-dimensional compact, convex sets. By the induction hypothesis, we have $q \in \operatorname{conv}\operatorname{ext}(F_q)$ és $r \in \operatorname{conv}\operatorname{ext}(F_r)$. But by the definition of face, there are linear functionals $f_q : \mathbb{R}^n \to \mathbb{R}$ and $f_r : \mathbb{R}^n \to \mathbb{R}$ attaining their minima exactly at F_q and F_r, respectively, and thus, by Theorem 3, the extremal points of F_q and F_r are extremal points of K. But then $p \in [q, r] \subseteq \operatorname{conv}(\operatorname{ext}(F_q) \cup \operatorname{ext}(F_r)) \subseteq \operatorname{conv}(\operatorname{ext}(K))$. □

We have seen that the extremal points of a set are not necessarily exposed points. On the other hand, it is true that they are accumulation points of sequences of exposed points.

Theorem 5 (Straszevicz)**.** *For any compact, convex set $K \subset \mathbb{R}^n$ we have $K = \mathrm{cl}(\mathrm{conv}(\mathrm{ex}(K)))$; or in other words, K is equal to the closure of convex hull of its exposed points.*

Proof. Let $x \in \mathrm{ext}(K)$ and $\varepsilon > 0$ be arbitrary. Let us consider the compact, convex set $K_\varepsilon = \mathrm{conv}(K \setminus \mathrm{int}\, B_\varepsilon(x)) \subseteq K$, where $B_\varepsilon(x)$ denotes the closed ball of radius ε and center x. If $x \in K_\varepsilon$, then by the Carathéodory theorem it is the convex combination of at most $n+1$ points of $(K \setminus \mathrm{int}\, B_\varepsilon(x))$; that is, it is a relative interior point of a segment in K. But this contradicts the assumption that $x \in \mathrm{ext}(K)$, and thus, $x \notin K_\varepsilon$.

Note that K_ε is a compact, convex set, and thus, it can be strictly separated from p. In other words, there is a hyperplane H such that one of the closed half spaces bounded by it intersects K in a subset of $B_\varepsilon(x)$, and this half space contains x in its interior. Let H^+ denote this closed half space. Let L be the half line starting at x, perpendicular to H and intersecting H. For any $y \in L$ let $z(y)$ be a farthest point of K from y. Then $z(y) \in \mathrm{ex}(K)$ for any $y \in L$ (see Problem sheet 5, Exercise 4). On the other hand, if y is sufficiently far from x, then $z(y) \in B_\varepsilon(x)$. Thus $x \in \mathrm{cl}(\mathrm{ex}(K))$, from which $\mathrm{ext}(K) \subseteq \mathrm{cl}(\mathrm{ex}(K))$.

By the containment relation $\mathrm{conv}(\mathrm{cl}(X)) \subseteq \mathrm{cl}(\mathrm{conv}(X))$, satisfied for any set $X \subseteq \mathbb{R}^n$, and by the Krein-Milman Theorem, we have

$$K \subseteq \mathrm{conv}(\mathrm{ext}(K)) \subseteq \mathrm{conv}(\mathrm{cl}(\mathrm{ex}(K))) \subseteq \mathrm{cl}(\mathrm{conv}(\mathrm{ex}(K))) \subseteq K,$$

that is, $K = \mathrm{cl}(\mathrm{conv}(\mathrm{ex}(K)))$. □