LECTURE 8: CONVEX POLYTOPES

Our next topic is the theory of convex polytopes. Our main concept is as follows.

Definition 1. *The convex hull of finitely many points in* $\mathbb{R}^n$ *is called a* convex polytope*, or shortly,* polytope*. If* $P \subset \mathbb{R}^n$ *is a convex polytope, then the set* $\{x_1, x_2, \ldots, x_k\} \subset \mathbb{R}^n$ *is a* minimal representation *of* P*, if*

(i) $P = \operatorname{conv}\{x_1, x_2, \ldots, x_k\}$*, and*
(ii) *for any index* i*, we have* $x_i \notin \operatorname{conv}\{x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_k\}$.

Let us observe that every convex polytope has a minimal representation, which can be obtained by removing redundant points one by one from any represention. It is worth noting that the exposed points (that is, 0-dimensional faces) of a convex polytope are usually called *vertices*, and the $(n-1)$-dimensional faces of a convex polytope are called *facets*.

Theorem 1. *Let* $M = \{x_1, \ldots, x_k\} \subset \mathbb{R}^n$ *be a minimal representation of the convex polytope* P*. Then the following are equivalent:*

(i) $x \in M$*,*
(ii) $x \in \operatorname{ex}(P)$*,*
(iii) $x \in \operatorname{ext}(P)$*.*

Proof. First, we show that (i) implies (ii). Assume that $x \in M$. Then $x \notin \operatorname{conv}(M \setminus \{x\})$. Since $\operatorname{conv}(M \setminus \{x\})$ is compact and convex, there is a hyperplane H that strictly separates it from x. Let H_0 be the hyperplane parallel to H and containing x. Then $H_0 \cap M = \{x\}$ and H_0 is a supporting hyperplane of $P = \operatorname{conv}(M)$. By Proposition 1 of the second lecture, then $H_0 \cap P = H_0 \cap \operatorname{conv}(M) = \operatorname{conv}(H_0 \cap M) = \{x\}$, and hence, x is a vertex of P.

By Proposition 2 of the fifth lecture, for any closed, convex set K we have $\operatorname{ex}(K) \subseteq \operatorname{ext}(K)$. As M is compact, so is its convex hull; that is, (ii) implies (iii). We will show that (iii) implies (i). Let $x \in \operatorname{ext}(P)$. Now, if $x \in \operatorname{conv}(M \setminus \{x\})$ was true, then x could be written as a convex combination of points from $M \setminus \{x\}$. Choosing a minimal number of such points one can show that then x could be written as a relative interior point of a segment in P, which would contradict the condition that $x \in \operatorname{ext}(P)$. □

Corollary 1. *Every convex polytope has a unique minimal representation.*

Remark 1. *By Proposition 1 of the second lecture, if H is a supporting hyperplane of the convex set* $\mathrm{conv}(X)$*, then* $H \cap \mathrm{conv}(X) = \mathrm{conv}(H \cap X)$*. From this it follows that every face of a convex polytope is a convex polytope, and also that every convex polytope has only finitely many faces.*

The next two statements hold for the faces of every compact, convex sets.

Proposition 1. *If $K \subset \mathbb{R}^n$ is a nonempty, compact, convex set, and $F_1, \ldots, F_m$ are faces of K, then $F = \bigcap_{i=1}^m F_i$ is a face of K.*

Proof. If $F = \emptyset$, then F is a face of K, and thus, we may assume that $F \neq \emptyset$, which implies that for every i, F_i is a proper face of K. Without loss of generality, we may assume that $o \in F$. Since F_i is a proper face of K, there is a linear functional $f_i : \mathbb{R}^n \to \mathbb{R}$ satisfying $f_i(x) \geq 0$ for all $x \in K$, and for which $f(x) = 0$ for some $x \in K$ if and only if $x \in F_i$. Now, let $f = \sum_{i=1}^m f_i$. This function f is a linear functional, and if $x \in K$, then $f(x) \geq 0$. Assume that $x \in K$ and $f(x) = 0$. From this, $\sum_{i=1}^m f_i(x) = 0$, but since $f_i(x) \geq 0$ for any value of i, this is satisfied if and only if $f_i(x) = 0$ for all values of i, or in other words, if $x \in F$. Thus, F is a face of K. □

Proposition 2. *Let $S_2 \subseteq S_1 \subset \mathbb{R}^n$ be compact, convex sets. If F is a face of S_1, then $F \cap S_2$ is a face of S_2.*

Proof. If $F \cap S_2 = \emptyset$, then it is clearly a face of S_2. Assume that $F \cap S_2 \neq \emptyset$, which implies that F is a proper face of S_1. Let H be a supporting hyperplane of S_1 satisfying $H \cap S_1 = F$. Then H also supports S_2, and $H \cap S_2 = (H \cap S_1) \cap S_2 = F \cap S_2$, implying that $F \cap S_2$ is a face of S_2. □

Our next proposition, which, in some sense, is the converse of the previous one, holds only for convex polytopes.

Proposition 3. *Let F_1 be a proper face of a convex polytope P, and let F_2 be a face of F_1. Then F_2 is a face of P.*

Proof. If $F_2 = \emptyset$, then the statement holds, and hence, we may assume that F_2 is a proper face of F_1. According to our conditions, P has a supporting hyperplane H in $\mathbb{R}^n$ satisfying $P \cap H = F_1$, and if F_2 is a proper face of F_1, then there is a 'supporting hyperplane' G of F_2 in H satisfying $G \cap F_1 = F_2$. Observe that $\dim G = n - 2$. As P is a convex polytope, only finitely many vertices of P are not elements of H, and thus, H can be rotated around G with a sufficiently small angle in a suitable direction such that the hyperplane H' obtained by this rotation is a supporting hyperplane of P, and, from amongst the vertices of P, it contains only those in F_2. But from this, it follows that $H' \cap P = F_2$, yielding that F_2 is a face of P. $\square$

Problem 1. *Construct a compact, convex set $K \subseteq \mathbb{R}^n$ with the property that it has a proper face F_1, and F_1 has a proper face F_2 such that F_2 is not a face of K.*

We have seen that every compact, convex set can be obtained as the intersection of closed half spaces. Now we show that a convex polytope is the intersection of finitely many closed half spaces (namely those defined by its facets).

Definition 2. *The intersection of finitely many closed half spaces is called a* polyhedral set.

Theorem 2. *Every convex polytope is a bounded polyhedral set.*

Proof. Let $P \subset \mathbb{R}^n$ be a convex polytope. As P is compact, it is sufficient to prove that it is a polyhedral set. Without loss of generality, assume that $\dim P = n$, as every hyperplane is obtained as the intersection of the two closed half spaces it generates, and every affine subspace is obtained as the intersection of finitely many hyperplanes.

Let $M = \{x_1, \ldots, x_k\}$ be a minimal representation of P. Let the facets of P be $F_1, \ldots, F_m$, and denote by H_i and H_i^+ the supporting hyperplane and the closed supporting half space defined by F_i, respectively. Then for any index i, we have $P \cap H_i = F_i$ and $P \subset H_i^+$. We show that $P = \bigcap_{i=1}^m H_i^+$.

Cearly, $P \subseteq \bigcap_{i=1}^m H_i^+$, and thus, by contradiction, we suppose that there is a point $x \in \left(\bigcap_{i=1}^m H_i^+ \right) \setminus P$. Now, let $D = \bigcup \operatorname{aff}(\{x\} \cup C)$, where C runs over the family of the subsets of M of cardinality at most $(n-1)$. Then D is the union of finitely many affine subspaces of dimension at most $(n-1)$, and thus, we can choose a point $y \in \operatorname{int}(P)$ with $y \notin D$. But then, by $x \notin P$, the segment $[x, y]$ intersects the boundary of P, that is, there is a point $z \in (x, y)$ with $z \in \operatorname{bd}(P)$. We will show that z lies on a facet of P, but it does not lie on any lower dimensional face of P.

Assume that $z \in F$ for some j-dimensional face of P, where $0 \leq j \leq n-2$. Then, by Carathéodory's theorem, z is contained in the convex hull of at most $(n-1)$ points of M, implying $\operatorname{aff}\{x, z\} \in D$, which contradicts the assumption that $y \notin D$. By Corollary 4 of the first lecture, any boundary point of a compact, convex set is a point of a supporting hyperplane of the set, and thus, a point of a proper face of the set. Thus, by exclusion, z is a point of a facet F_i of P. But from this, by $y \in \operatorname{int} P \subset \operatorname{int} H_i^+$, we obtain $x \notin H_i$, which contradicts our choice of y. This yields that $P = \bigcap_{i=1}^m H_i^+$. □

Corollary 2. *The boundary of every n-dimensional convex polytope $P \subset \mathbb{R}^n$ is the union of the facets of P.*