Convex Geometry
For students with mathematics major

Practice problems for the 1st midterm

Exercise 1. Consider an A $(n \times n)$ matrix as a point of the space $\mathbb{R}^{n^2}$. Let $\mathcal{S}$, $\mathcal{S}_+$ and $\mathcal{S}_{++}$ the set of $(n \times n)$ symmetric, positive definit and positive semidefinit matrices in $\mathbb{R}^{n^2}$, respectively. Prove that these sets are convex.

Exercise 2. Let $L, L' \subset \mathbb{R}^n$ be linear subspaces of dimensions k and $n-k$, intersecting only in o, where $0 \leq k \leq n$.

a) Prove that then for any $p, q \in \mathbb{R}^n$, the affine subspaces $p + L$ and $q + L'$ intersect in a unique point.

b) Let $F = p + L$ fixed. Define the function $\pi : \mathbb{R}^n \to F$ as $\pi(q) = q'$, where $F \cap (q + L') = \{q'\}$ (the map π is called *projection onto F parallel to L'*). Prove that for any convex set $K \subseteq \mathbb{R}^n$ the set $\pi(K)$ is convex, and for any convex set $K' \subseteq F$ the set $\pi^{-1}(K')$ is convex.

Exercise 3. Prove that if $K \subseteq \mathbb{R}^n$ is convex, closed and unbounded, then for every point of K there is a closed half line in K starting at this point.

Exercise 4. Let $K \subseteq \mathbb{R}^n$ be a closed, convex set. Prove that if K contains a closed half line starting at p, then for every $q \in K$ it contains the half line starting at q and having the same direction as that of L.

Exercise 5. Let $K \subseteq \mathbb{R}^n$ the convex hull of the points with all their coordinates equal to 1 or -1 (that is, K is an o-symmetric n-dimensional cube with edge length 2). Construct K as an intersection of closed half spaces.

Exercise 6. Let $e_1, \ldots, e_n$ be the vectors of the usual orthonormal basis of the space $\mathbb{R}^n$. Let $K = \operatorname{conv}\{\pm e_1, \pm e_2, \ldots, \pm e_n\}$ (that is, K is an o-symmetric, regular cross-polytope). Construct K as an intersection of closed half spaces.

Exercise 7. Let $X \subset \mathbb{R}^n$ be a compact set. Let H be an open half space that contains X. Show that then there is a closed half space $H_0 \subset H$ that contains X. Is the statement true if X is not necessarily closed?