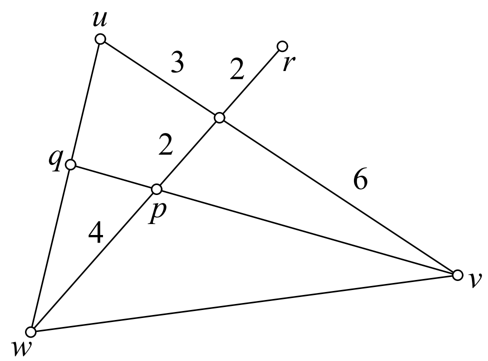
Convex Geometry tutorial
For students with mathematics major

Problem sheet 1 - Affine and convex combinations - Solutions

Exercise 1. In the picture numbers denote lengths of segments. Express the points p, q, r as affine combinations of the points u, v, w.



Solution

Let x be the intersection point of the segment $[u, v]$ and the straight line through $[p, w]$. By the formula for the point dividing a segment in a given ratio we have $x = \frac{2}{3}u + \frac{1}{3}v$. By the same formula we obtain $p = \frac{2}{3}x + \frac{1}{3}w = \frac{4}{9}u + \frac{2}{9}v + \frac{1}{3}w$ and $p = \frac{1}{2}w + \frac{1}{2}r$, from which $r = 2p - w = \frac{8}{9}u + \frac{4}{9}v - \frac{1}{3}w$ follows. Finally, we know that q lies in the affine hulls of both the pair u, w and the pair p, v, implying that it can be written as an affine combination of both pairs. Since these pairs are affinely independent, these combinations are unique. Thus,

$$q = \alpha u + (1-\alpha)w = \beta p + (1-\beta)v = \beta\left(\frac{4}{9}u + \frac{2}{9}v + \frac{1}{3}w\right) + (1-\beta)v = \frac{4\beta}{9}u + \left(1 - \frac{7\beta}{9}\right)v + \frac{\beta}{3}w,$$

from which $\alpha = \frac{4\beta}{9}, 1 - \frac{7\beta}{9} = 0, (1-\alpha) = \frac{\beta}{3}$. Solving these we have $\alpha = \frac{4}{7}, \beta = \frac{9}{7}$, that is, $q = \frac{4}{7}u + \frac{3}{7}w$.

Exercise 2. Let F_1 and F_2 be affine subspaces of $\mathbb{R}^n$. Assume that $F_1 \cap F_2 \neq \emptyset$. Prove that then $\dim F_1 + \dim F_2 - n \leq \dim(F_1 \cap F_2)$.

Solution

Let $p \in F_1 \cap F_2$. Then there are linear subspaces L_1, L_2 in $\mathbb{R}^n$ for which $F_1 = p + L_1$ and $F_2 = p + L_2$. Let $a_1, \ldots, a_k$ be a basis of $L_1 \cap L_2$. Since any linearly independent vector system can be extended to a basis, there are some vectors $b_1, \ldots, b_{m_1} \in L_1$ and $c_1, \ldots, c_{m_2} \in L_2$ such that $a_1, \ldots, a_k, b_1, \ldots, b_{m_1}$ is a basis of L_1, and $a_1, \ldots, a_k, c_1, \ldots, c_{m_2}$ is a basis of L_2. We will show that these vectors are linearly independent. Indeed, if $\sum_{i=1}^k \alpha_i a_i + \sum_{i=1}^{m_1} \beta_i b_i + \sum_{i=1}^{m_2} \gamma_i c_i = o$ for suitable real numbers $\alpha_i, \beta_i, \gamma_i$, then by rearrangement we have

$$\sum_{i=1}^{k} \alpha_i a_i + \sum_{i=1}^{m_1} \beta_i b_i = -\sum_{i=1}^{m_2} \gamma_i c_i.$$

Observe that the left hand side is in F_1 and the right hand side is in F_2, implying that the vector is in $F_1 \cap F_2$, which yields that all coefficients β_i, γ_i are zero. But then $\sum_{i=1}^k \alpha_i a_i = o$, from which, by the choice of the vectors a_i, we have that all α_is are zero. Thus, the vectors

$a_1, \ldots, a_k, b_1, \ldots, b_{m_1}, c_1, \ldots, c_{m_2}$ are linearly independent in $\mathbb{R}^n$, from which $k + m_1 + m_2 \leq n$. This implies the statement.

Exercise 3. Let $K \subseteq \mathbb{R}^n$ be a convex set. Prove that $\operatorname{int} K$ and $\operatorname{cl} K$ are convex sets.

Solution

Let $B = \{x : ||x|| \leq 1\}$ denote the closed unit ball centered at the origin, and let $p, q \in\subseteq \operatorname{int}(K)$. Then there is some $\varepsilon > 0$ such that $p + \varepsilon B, q + \varepsilon B \subseteq K$. Consider some point $tp + (1-t)q$, where $t \in [0,1]$. We will show that $tp + (1-t)q + B \subseteq K$. Indeed, if $z \in B$ arbitrary, then $tp + (1-t)q + z = t(p+z) + (1-t)(q+z)$, where $p + z \in p + B \subset K$ and $q + z \in q + B \subset K$, and by the convexity of K we have $tp + (1-t)q + z \in K$. Thus, a neighborhood of the point $tp + (1-t)q$ belongs to K, from which $tp + (1-t)q \in \operatorname{int}(K)$.

Now, let $p, q \in\subseteq \operatorname{cl}(K)$, and $t \in [0,1]$. We will show that $tp + (1-t)q \in \operatorname{cl}(K)$. By the definition of closure there are sequences p_m, q_m in K for which $p_m \to p$ és $q_m \to q$. But the convexity of K yields $tp_m + (1-t)q_m \in K$ for every m. On the other hand, the continuity of vector addition and multiplication by a scalar yields $tp_m + (1-t)q_m \to tp + (1-t)q$, from which $tp + (1-t)q \in \operatorname{cl}(K)$ follows.

Exercise 4. Verify that the intersection of arbitrarily many convex sets is convex.

Solution

Let I be an index set, and $K_i, i \in I$ convex sets. Let $p, q \in \bigcap_{i \in I} K_i$. Then $p, q \in K_i$ for every index i, from which the convexity of K_i implies $[p,q] \subseteq K_i$; that is, $[p,q] \cap_{i \in I} K_i$. Thus, $\bigcap_{i \in I} K_i$ is convex.

Exercise 5. Let $x_1, \ldots, x_k \in \mathbb{R}^n$, és $\alpha_1, \alpha_2, \ldots, \alpha_k \in \mathbb{R}$. Prove that the set

$$P = \{y \in \mathbb{R}^n : \langle y, x_i \rangle \leq \alpha_i, i = 1, 2, 3 \ldots, k\}$$

is convex. Is it true in case of infinitely many inequalities?

Solution

Let $x \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$. We show that $X = \{y \in \mathbb{R}^n : \langle x, y \rangle \leq \alpha\}$ is convex. Indeed, if $p, q \in X$, then $\langle x, p \rangle \leq \alpha$ and $\langle x, q \rangle \leq \alpha$. But then for every $t \in [0,1]$

$$\langle x, tp + (1-t)q \rangle = t\langle x, p \rangle + (1-t)\langle x, q \rangle \leq t\alpha + (1-t)\alpha = \alpha,$$

implying $tp + (1-t)q \in X$. But

$$P = \bigcup_{i=1}^{k} \{y \in \mathbb{R}^n : \langle y, x_i \rangle \leq \alpha_i\},$$

and thus, P is the intersection of convex sets, and therefore it is convex. This argument can be applied also when the number of inequalities is infinite.

Exercise 6. Let $S \subseteq \mathbb{R}^n$ be arbitrary. Let the *kernel* of S be the set of points x with the property that $[x, y] \subseteq S$ holds for any $y \in S$. Prove that the kernel of S is convex.

Solution

Assume that p, q are points of the kernel of S. We need to prove that every point $r \in [p,q]$ belongs to the kernel of S; or in other words, for any point $y \in S$ we have $[r, y] \subseteq S$. Let $z \in [r, y]$. By it choice, z is a point of the triangle with vertices p, q, y; that is, there is some $z_0 \in [q, y]$ such that $z \in [p, z_0]$. But since p, q are points of the kernel of S, we have $z_0 \in S$ implying $z \in S$ and thus $[r, y] \subseteq S$. From this the convexity of the kernel of S readily follows.

Exercise 7. Let $K \subseteq \mathbb{R}^n$ be convex, and let $T : \mathbb{R}^n \to \mathbb{R}^n$ be an invertible linear transformation. Prove that the set $T(K) = \{T(x) : x \in K\}$ is convex. Prove that for the set P defined in Exercise 5 there are vectors $w_1, w_2, \ldots, w_k \in \mathbb{R}^n$ and numbers $\beta_1, \beta_2, \ldots, \beta_k \in \mathbb{R}$ such that

$$T(P) = \{y \in \mathbb{R}^n : \langle y, w_i \rangle \leq \beta_k, i = 1, 2, 2 \ldots, k\}.$$

Solution

Let $y_1, y_2 \in T(K)$. We will show that $[y_1, y_2] \subset T(K)$. Let us choose points x_1, x_2 for which $T(x_1) = y_1$ and $T(x_2) = y_2$. If $t \in [0,1]$, then the linearity of T implies $T(tx_1 + (1-t)x_2) = tT(x_1) + (1-t)T(x_2) = ty_1 + (1-t)y_2 \in T(K)$, which yields that $T(K)$ is convex.

To prove the second statement, it is sufficient to show that for any $x \in \mathbb{R}^n$, $\alpha \in \mathbb{R}$, the set $\{T(y) : \langle x, y \rangle \leq \alpha\}$ can be written in the form $\{z : \langle w, z \rangle \leq \beta\}$ for suitable $w \in \mathbb{R}^n$ $\beta \in \mathbb{R}$. But

$$\{T(y) : \langle x, y \rangle \leq \alpha\} = \{z : \langle x, T^{-1}(z) \rangle \leq \alpha\} = \{z : \langle \left(T^{-1}\right)^T (x), z \rangle \leq \alpha\},$$

and hence, the statement follows with $w = \left(T^{-1}\right)^T (x)$ and $\beta = \alpha$.