Convex Geometry tutorial
For students with mathematics major

Problem sheet 3 - Helly's theorem - Solutions

Exercise 1. Show that the finite version of Helly's theorem does not hold for nonconvex sets.

Solution.

Let k be an arbitrary positive integer, $A = \{p_1, \ldots, p_k\} \subset \mathbb{R}^n$ be an arbitrary point set, and $A_k = A \setminus \{p_k\}$. Then, removing any of the sets A_k, the intersetion of the rest is nonempty, but the intersection of all sets is empty.

Exercise 2. Give an example of a family of infinitely many closed, convex sets in $\mathbb{R}^n$ with the property that any $n+1$ elements of the set intersect, but the intersection of all elements is empty.

Solution.

Let $A_k = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_1 \geq k\}$, where k is an arbitrary positive integer. Then the sets A_k have an empty intersection, but any finitely many of them have a nonempty intersection.

Exercise 3. Give an example of a family of finitely many convex sets in the plane such that no two of them are disjoint, but the intersection of all of them is empty.

Solution.

Let this family consist of finitely many mutually non-parallel straight lines.

Exercise 4. Let $\mathcal{F}$ be a finite family of convex sets in $\mathbb{R}^n$, and let L be an affine subspace in $\mathbb{R}^n$. Prove that if for any at most $n+1$ elements of $\mathcal{F}$ a translate of L intersects all elements, then there is a translate of L that intersects all elements of $\mathcal{F}$.

Solution.

As affine subspaces are convex, this exercise is a special case of the part of the next exercise about intersections.

Exercise 5. Let $\mathcal{F}$ be a finite family of convex sets and let C be a convex set in $\mathbb{R}^n$. Prove that if, for any at most $(n+1)$ elements of $\mathcal{F}$, there is a translate of C that intersects/contains/is contained in all of them, then then there is a translate of C that intersects/contains/is contained in all elements of $\mathcal{F}$.

Solution.

First, we show the statement for intersection. Let $\mathcal{F} = \{K_1, K_2, \ldots, K_m\}$ be a finite family of convex sets in $\mathbb{R}^n$, and let $C \subseteq \mathbb{R}^n$ be a convex set such that a suitable translate of C intersects any at most $n+1$ elements of $\mathcal{F}$. For every subscript i let X_i denote the set of translation vectors $x \in \mathbb{R}^n$ such that $x + C$ intersects K_i, that is,

$$X_i := \{x \in \mathbb{R}^n : (x + C) \cap K_i \neq \emptyset\}.$$

According to the conditions, for any $1 \leq i_1, i_2, \ldots, i_{n+1} \leq m$ (not necessarily distinct) indices, we have $\bigcap_{j=1}^{n+1} X_{i_j} \neq \emptyset$. We show that X_i is convex for every i. Let $x_1, x_2 \in X_i$. Then, by our conditions, $(x_1 + C) \cap K_i \neq \emptyset \neq (x_2 + C) \cap K_i$, that is, there are some $y_1, y_2 \in C$ such that $x_1 + y_1, x_2 + y_2 \in K_i$. But as C and K_i are convex, for any $t \in [0,1]$ we have $ty_1 + (1-t)y_2 \in C$ és $tx_1 + (1-t)x_2 + ty_1 + (1-t)y_2 \in K_i$; that is, $tx_1 + (1-t)x_2 + ty_1 + (1-t)y_2 \in (tx_1 + (1-t)x_2 + C) \cap K_i \neq \emptyset$, from which it follows that $tx_1 + (1-t)x_2 \in X_i$, and hence, X_i is convex. Thus, we can apply the finite version of Helly's theorem for the convex sets X_i, which yields $\bigcap_{i=1}^{m} X_i \neq \emptyset$.

In the other two cases we can apply a similar consideration. Indeed, let

$$Y_i =:= \{x \in \mathbb{R}^n : (x + C) \subseteq K_i\}, \quad \text{and} \quad Z_i =:= \{x \in \mathbb{R}^n : K_i \subseteq (x + C)\}.$$

If for any at most $n+1$ elements of $\mathcal{F}$ a translate of C contains all of them, then for any $1 \leq i_1, i_2, \ldots, i_{n+1} \leq m$ (not necessarily distinct) indices $\bigcap_{j=1}^{n+1} Y_{i_j} \neq \emptyset$ ($\bigcap_{j=1}^{n+1} Z_{i_j} \neq \emptyset$) is satisfied. Thus, it is sufficient to prove that Y_i and Z_i are convex.

If $x_1, x_2 \in Y_i$, then for every point $y \in C$ we have $x_1 + y, x_2 + y \in K_i$, and therefore by the convexity of K_i for any $t \in [0,1]$ the containment relation $t(x_1 + y) + (1-t)(x_2 + y) = tx_1 + (1-t)x_2 + y \in K_i$ is satisfied, yielding $tx_1 + (1-t)x_2 \in Y_i$ and the convexity of Y_i. Similarly, if $x_1, x_2 \in Z_i$, then for every point $y \in K_i$ we have $x_1 + y, x_2 + y \in C$, and thus by the convexity of C, for every $t \in [0,1]$ we have $t(x_1 + y) + (1-t)(x_2 + y) = tx_1 + (1-t)x_2 + y \in C$, implying $tx_1 + (1-t)x_2 \in Z_i$ and the convexity of Z_i. Thus, the second and the third statement also follows from the the finite version of Helly's theorem.

Exercise 6. *(Krasnosselsky's art gallery theorem) Let $S \subset \mathbb{R}^n$ be a compact set of at least $n+1$ points, and assume that for any $p_1, p_2, \ldots, p_{n+1}$ there is some $q \in S$ from which every p_i is visible, or in other words, $[p_1, q], \ldots, [p_{n+1}, q] \subseteq S$. Prove that then S is *starlike*, that is, it contains a point from which every point of S is visible.

Solution.

In the solution we denote by $\mathrm{d}(A, B)$ the distance of the sets $A, B \subset \mathbb{R}^n$, defined as $\inf\{||a - b|| : a \in A, b \in B\}$.

For any point $x \in S$, let V_x denote the set of the points of S visibel from x, that is, if $q \in V_x$, then $[q, x] \subseteq S$. As S is compact, V_x is compact for any x, implying that also $\mathrm{conv}(V_x)$ is compact. Thus, we can apply the infinite version of Helly's theorem, which yields that there is some $y \in \bigcap_{x \in S} \mathrm{conv}(V_x)$.

By contradiction, assume that $y \notin V_x$ for some point $x \in S$, that is, there is some $z \in [x, y]$, $z \notin S$. By the compactness of S there is a point x' on this segment closest to x in S. Let x'' the point of $[x', z]$ satisfying $||x'' - x'|| = \frac{1}{2}\mathrm{d}(\{z\}, S)$. Since $[x'', z]$ and S are disjoint, compact sets, there is some $u \in [x'', z]$ and $v \in S$ for which $||u - v|| = \mathrm{d}([x'', z], S) > 0$. Let H_v denote the closed half space not containing u, and bounded by the hyperplane through v and perpendicular to $[u, v]$. Then, by the choice of u, v, $S \subset H_v$. But then $\mathrm{conv}\, V_v \subset H_v$, and hence $y \in H_v$ is also satisfied. On the other hand, $x \in S$ implies $x \in H_v$, and thus, $[x, y] \subset H_v$, which contradicts the choice of u.