Convex Geometry tutorial
for students with mathematics major

Problem sheet 4 - Hyperplanes, Minkowski sum, separation - Solutions

Exercise 1. Prove that if A and B are two disjoint, convex sets in $\mathbb{R}^n$, then there are disjoint convex sets A', B' in $\mathbb{R}^n$ satisfying $A \subset A'$, $B \subset B'$, and $A' \cup B' = \mathbb{R}^n$.

Solution

We prove the statement by induction on n. Let $n = 1$. If, e.g. $L = \emptyset$, then $K^* = \mathbb{R}^1$ and $L^* = \emptyset$ satisfy the required conditions. If K and L are not empty, then by the Separation Theorem there is a point p such that hogy one of the two closed half lines starting at p contains K, and the other one contains L. Let these two half lines be E_1 and E_2 such that $K \subseteq E_1$ and $L \subseteq E_2$. Since K and L are disjoint, we can assume that e.g. $p \notin L$. Then $K^* = E_1$ and $L^* = E_2 \setminus \{p\}$ satisfy the required conditions.

Now, assume that the statement holds in $\mathbb{R}^k$ for any $k < n$. If e.g. $L = \emptyset$, then $K^* = \mathbb{R}^n$ és $L^* = \emptyset$ satisfy the conditions. In the opposite case, as K and L are disjoint, convex sets, there is a hyperplane H that (not necessarily strictly) separates K and L. Let the two open half spaces bounded by H be denoted by H_+ and H_-, where $K \subseteq H \cup H_+$ and $L \subseteq H \cup H_-$. The sets $H \cap K$ and $H \cap L$ are convex, being the intersections of convex sets, and they are disjoint. Since H is an $(n-1)$-dimensional Euclidean space, we may apply the induction hypothesis, and obtain that there are disjoint, convex sets K_H^* and L_H^*, for which $H \cap K \subseteq K_H^*$, $H \cap L \subseteq L_H^*$, and $H = K_H^* \cup L_H^*$. Let $K^* = H_+ \cup K_H^*$ and $L^* = H_- \cup L_H^*$. These sets are disjoint, and their union is $\mathbb{R}^n$. We show that they are convex.

CLearly, it is sufficient to show that K^* is convex. Let $p, q \in K^*$. If $p, q \in K_H^*$, then by the convexity of K_H^* we have $[p, q] \subseteq K_H^* \subset K^*$. Let, e.g. $p \in H_+$. Then for any point $r \in [p, q]$, $r \neq q$ we have $r \in H_1 \subset K^*$, therefore $[p, q] \subseteq K^*$, implying that K^* is convex. Thus, K^* and L^* satisfy the conditions of the exercise.

Exercise 2. Describe all decompositions of the 3-dimensional Euclidean space into the union of two disjoint, convex sets. What is the situation in $\mathbb{R}^n$?

Solution

Let us call a set X in $\mathbb{R}^n$ a *suitable set*, if $X = \emptyset$, $X = \mathbb{R}^n$, or for some $1 \leq k \leq n$ there are sets $H_+^1, H_+^2, \ldots, H_+^k$, H^k such that

(i) H_+^1 is an open half space in $\mathbb{R}^n$, H_+^2 is an open half space in the boundary of H_+^1, and in general, for every $2 \leq m \leq k$, H_+^m is an open half space in the relative boundary of H_+^{m-1},

(ii) H^k is the relative boundary of H_+^k,

(iii) $X = \bigcup_{i=1}^k H_+^i$ (first type suitable set), or $X = H_k \cup \bigcup_{i=1}^k H_+^i$ (second type suitable set).

by the argument used in the previous exercise, every suitable set is convex, and the complement of a first type suitable set is a second type suitable set, and vice versa. Therefore any decomposition of $\mathbb{R}^n$ into the union of two suitable sets satisfies the conditions. We show that if K and L are convex sets whose disjoint union is $\mathbb{R}^n$, then K and L are suitable sets.

We prove the statement by induction on n. We may assume that K and L are not empty. Applying the idea of the solution of Exercise 1, we have that in this case one of K and L is a closed half line and the other one is an open half line, which are suitable sets. Assume that the statement is true for the decompositions of $\mathbb{R}^{n-1}$, and let K, L be disjoint, convex sets satisfying $K \cup L = \mathbb{R}^n$.

The sets $\emptyset$, $\mathbb{R}^n$ satisfy the conditions. If K and L are not this pair, then by the Separation Theorem there is a hyperplane H^1 separating K and L, and thus, suitably labelling the open half planes H^1_+, H^1_- bounded by H^1, we have $K \subseteq H^1 \cup H^1_+$ and $L \subseteq H^1 \cup H^1_-$, from which $H^1_+ \subseteq K$ and $H^1_- \subseteq L$. Let $K^1 = K \cap H^1$ and $L^1 = L \cap H^1$. Then $K = K^1 \cup H^1_+$ and $L = L^1 \cup H^1_-$. On the other hand, K^1 and L^1 are disjoint, convex sets, and $K^1 \cup L^1 = H^1$, which, by the induction hypothesis, implies that K^1 and L^1 are suitable sets in H^1. But then K and L are suitable sets in $\mathbb{R}^n$.

Exercise 3. (a) Let T be a regular triangle. What is $T - T$? What is $T + T$?

(b) For any compact set $T \subset \mathbb{R}^n$ and positive integer k let $T_k = \frac{\overbrace{T+T+\ldots+T}^{k}}{k}$. Prove that if T is convex and $k \in \mathbb{Z}^+$, then

$$T_k = T.$$

If T is not necessarily convex, what is the relationship between T, T_k and $\operatorname{conv}(T)$?

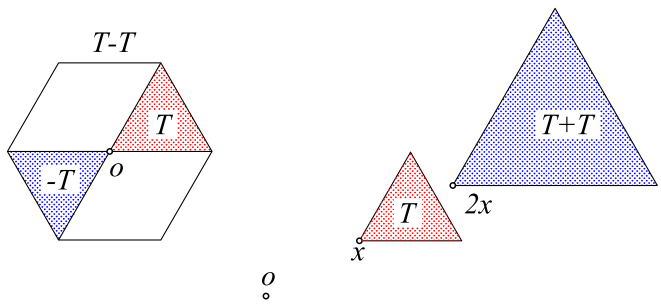
(c)* Prove that if $T \subset \mathbb{R}^n$ is compact and convex and$k \in \mathbb{Z}^+$, then

$$V(T_k) \leq V(T_{k+1}),$$

where $V(\cdot)$ denotes n-dimensional volume (Lebesgue measure). What happens if T is not necessarily convex?

Solution

a) Observe that for any $A, B \subset \mathbb{R}^n$ and $x, y \in \mathbb{R}^n$ we have $(x + A) + (y + B) = (x + y) + (A + B)$, that is, the vector sum of translates of two given sets is a translate of the vector sum of the sets. Thus, up to translation, $T - T$ is independent of the choice of the origin, and we may assume that one vertex of T is o, implying that $-T$ is the reflection of T about that vertex. On the other hand, for any sets $A, B \subset \mathbb{R}^n$, $A + B = \bigcup_{a \in A}(\{a\} + B)$, that is, $T - T$ can be obtained by sliding $-T$ along all points of T, and taking the union of all these translates of $-T$. In this waay one can see that $T - T$ is a regular hexagon centered at o, with edge length equal to the edge length of T and $-T$, and containing one of the edges of both these triangles in its boundary. We obtain similarly that if o is a vertex of T, then $T + T = 2T$, that is, $T + T$ coincides with the image of T under the central similarity of center o and ratio 2. If it is not true, then T can be written in the form $T = x + T_0$, where a vertex of T_0 is o. But then $T + T = (x + T_0) + (x + T_0) = 2x + (T_0 + T_0) = 2x + 2T_0 = 2T$, and the previous statement holds also in the general case, that is, $T + T$ coincides with the image of T under the central similarity of center o and ratio 2. In other words, $T + T = 2T$.

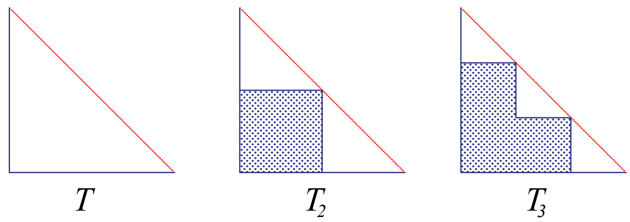


b) By the definition of Minkowski sum,

$$2T = \{x + x : x \in T\} \subseteq \{x + y : x, y \in T\} = T + T$$

for any nonempty set, from which $T \subseteq T_2$ follows. One can see similarly that $T \subseteq T_k$ for any nonempty set T and integer $k \in \mathbb{Z}^+$. On the other hand, since the elements of T_k are convex combinations of points of T, we also have $T_k \subseteq \text{conv}(T)$. If T is convex, then $T = \text{conv}(T)$, implying that then $T_k = T$ for any positive integer k.

One can ask whether the relation $T_k \subseteq T_m$ also holds for any positive integers $k \leq m$, even if T is not necessarily convex. This is not true in general, as shown by the set $T = [o, e_1] \cup [o, e_2] \subset \mathbb{R}^2$, where e_1, e_2 is the usual basis of the plane. Then, e.g. $T_2 \not\subseteq T_3$, as shown in the figure.



c) One can present the above example in a more general way. Let K_1 and K_2 be n-dimensional unit cubes in two orthogonal n-dimensional linear subspaces of $\mathbb{R}^{2n}$, and let $S = K_1 \cup K_2$. Let $S[k] = \overbrace{S + \ldots + S}^{k}$. Then

$$S[2] = S + S = (K_1 + K_1) \cup (K_1 + K_2) \cup (K_2 + K_1) \cup (K_2 + K_2) = (2K_1) \cup (K_1 + K_2) \cup (2K_2),$$

implying $\text{vol}(S[2]) = 1$. Similarly, $S[3] = \bigcup_{i=0}^{3}(iK_1 + (k-i)K_2)$. Since $(2K_1 + K_2) \cap (K_1 + 2K_2) = K_1 + K_2$, we have $\text{vol}(S[3]) = 2 \cdot 2^n - 1$. But this implies $\text{vol}(S[3]/3) = \frac{2^{n+1}-1}{3^{2n}} < \text{vol}(S[2]) = \frac{1}{2^{2n}}$, if n is sufficiently large.

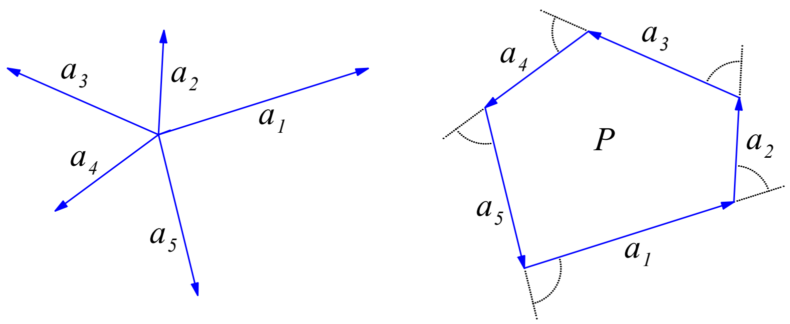
Exercise 4. Let the sum of the planar vectors $a_1, a_2, \ldots, a_k$ be o. Assume that among these vectors there are no two with the same direction. Prove that up to translation there is a unique convex polygon whose sides, oriented according to a fixed orientation of the plane, are exactly these vectors.

Solution

Draw these vectors in such a way that their starting points are o, and label them such that $a_1, a_2, \ldots, a_k$ are exactly in this order around o in counterclockwise order. Let P be an arbitrary convex polygon whose side vectors (oriented in counterclockwise order) are $a_1, a_2, \ldots, a_k$ in counterclockwise order. Let us walk around on the boundary of P in counterclockwise order. Then at every vertex we turn to the left with an angle between 0 and π (this angle is called *turning angle*). In case of a convex polygon, the sum of these turning angles is 2π, and thus, the side vectors of P are $a_1, a_2, \ldots, a_k$ if and only if they appear in this order in the boundary of P, that is, up to translation, P is unique.

Now, let Q be the polygonal curve obtained by connecting the points o, a_1, $a_1 + a_2, \ldots, a_1 + \ldots + a_k$ in this order.Since $\sum_{i=1}^{k} a_i = o$, Q is a closed polygonal curve. Let L_i be the line through the points $a_1 + \ldots + a_{i-1}$ and $a_1 + \ldots + a_i$.

We show that L_i is a sideline of Q, that is, one of the two closed half planes bounded by L_i contains Q. Since the labelling of the points is cyclic, it is sufficient to show it for the line L_1 containing the segment $[0, a_1]$. Using a suitable coordinate system we may assume that $a_1 = (0, x_1)$ for some suitable $x_1 > 0$. Let $a_i = (x_i, y_i)$, $x_i, y_i \in \mathbb{R}$ for all values of i. By the choice of the indices, there is some index $1 < m < k$ such that $y_1, \ldots y_m \geq 0$, and $y_{m+1}, \ldots, y_k < 0$. But the y-coordinates of the vertices of Q are $0, y_1 = 0, y_1 + y_2, y_1 + y_2 + y_3, \ldots, y_1 + \ldots + y_k = 0$ in this order, and thus, this sequence is increasing up to $y_1 + \ldots + y_m$, and decreasing after that, which implies that all vertices of Q, and also Q, are contained in the closed half space $\{(x, y) \in \mathbb{R}^2 : y \geq 0\}$.



Exercise 5. Let K and L be convex polygons, whose edge vectors, according to a fixed orientation of the plane, are $a_1, \ldots, a_k$ and $b_1, \ldots, b_m$, respectively. Prove that if, among the vectors, there are no two in the same direction, then $K + L$ is a convex polygon whose edge vectors are exactly $a_1, \ldots, a_k, b_1, \ldots, b_m$. How can we modify the statement if there are vectors with the same direction?

Solution

We use the notation of the previous exercise, and assume that the vectors a_i and b_j are labelled in such a way that starting with the direction of the positive half of the x-axis, their order according to positive (counterclockwise) orientation is $a_1, a_2, \ldots, a_k$, and $b_1, b_2, \ldots, b_m$, respectively. In addition, let $c_1, c_2, \ldots, c_{k+m}$ be the vector system, ordered according to positive orientation starting with the positive half of the x-axis, satisfying $\{c_1, \ldots, c_{k+m}\} = \{a_1, \ldots a_k\} \cup \{b_1, \ldots, b_m\}$. Let M be the convex polygon whose vertices are $o, c_1, c_1 + c_2, \ldots, c_1 + \ldots + c_{k+m} = o$ in this order. We show that $M = K + L$.

Observe that every vertex of M can be written in the form $(a_1 + \ldots + a_i) + (b_1 + \ldots + b_j)$ for some suitable indices i and j, and hence, every vertex of M is the sum of a vertex of K and a vertex of L, which implies $M \subseteq K + L$. We show that the line E through $[o, c_1]$ is a supporting line of $K + L$. To do this, we may assume that $c_1 = a_1$, and E coincides with the x-axis. Then, by the consideration in the previous problem, it follows that the y-coordinates of every vertex of K and L is nonnegative. We can say the same for all the convex combinations of these vertices, which implies that E is a supporting line of $K + L$. Similarly, one can see that every sideline (and similarly every supporting line) of M is a supporting line of $K + L$. Since a compact, convex set can be written as the intersection of its supporting half planes, from this $M = K + L$ follows.

If among the vectors there are some in the same direction, then the statement also holds, which can be shown, e.g. by a limit argument.

Definition. *If $K \subset \mathbb{R}^n$ is compact, convex, and $\operatorname{int} K \neq \emptyset$, then we say that K is a convex body. The* perimeter *of a plane convex body K is the supremum of the perimeters of the convex polygons contained in K, if it exists. Its notation:* $\operatorname{perim}(K)$.

Remark. *It can be shown that every plane convex body has a perimeter, and if $K \subseteq L$ are plane convex bodies, then* $\operatorname{perim}(K) \leq \operatorname{perim}(L)$.

Exercise 6. Let K and L be plane convex bodies. Prove that then $\operatorname{perim}(K + L) = \operatorname{perim}(K) + \operatorname{perim}(L)$.

Solution. Let $\varepsilon > 0$ be arbitrary. By the definition of perimeter, there are convex polygons $P \subseteq K$, $Q \subseteq L$ satisfying $0 \leq \operatorname{perim}(K) - \operatorname{perim}(P) < \frac{\varepsilon}{2}$ and $0 \leq \operatorname{perim}(L) - \operatorname{perim}(Q) < \frac{\varepsilon}{2}$. Then, by the properties of vector sum, we have $P+Q \subseteq K+L$, implying $\mathrm{perim}(P+Q) \subseteq \mathrm{perim}(K+L)$. But, by the previous exercise, $\mathrm{perim}(P+Q) = \mathrm{perim}(P) + \mathrm{perim}(Q)$, and thus, $\mathrm{perim}(K+L) > \mathrm{perim}(K) + \mathrm{perim}(L) - \varepsilon$ for any $\varepsilon > 0$, which implies $\mathrm{perim}(K+L) \geq \mathrm{perim}(K) + \mathrm{perim}(L)$.

On the other hand, let $X \subset K+L$ be a convex polygon satisfying $\mathrm{perim}(X) > \mathrm{perim}(K+L) - \varepsilon$. Let $X = \mathrm{conv}\{x_i + y_i : x_i \in K, y_i \in L, i = 1, 2, \dots, m\}$. Let $P = \mathrm{conv}\{x_i : i = 1, \dots, m\} \subseteq K$ and $Q = \mathrm{conv}\{y_i : i = 1, 2, \dots, m\} \subseteq L$. Then $X \subseteq P + Q \subseteq K + L$. By the definition of perimeter, $X \subseteq P + Q$ implies $\mathrm{perim}(X) \leq \mathrm{perim}(P+Q) = \mathrm{perim}(P) + \mathrm{perim}(Q) \leq \mathrm{perim}(K) + \mathrm{perim}(L)$, which yields $\mathrm{perim}(K) + \mathrm{perim}(L) \geq \mathrm{perim}(K+L) - \varepsilon$ for any $\varepsilon > 0$. Hence, $\mathrm{perim}(K) + \mathrm{perim}(L) \geq \mathrm{perim}(K+L)$. Comparing it to the inequality at the end of the previous paragraph, we obtain the desired equality.