Convex Geometry tutorial
for students with mathematics major

Problem sheet 5 - Supporting hyperplanes, faces of convex sets, extremal and exposed points, the Krein-Milman Theorem

Exercise 1. Prove that any compact, convex set in $\mathbb{R}^n$ can be written as the intersection of closed balls.

Solution

Let $K \subset \mathbb{R}^n$ be an arbitrary compact, convex set. We need to show that if $p \notin K$, then there is some closed ball $B_r(x)$ centered at x and with radius r such that $K \subseteq B_r(x)$ and $p \notin B_r(x)$, as this yields that the intersection of the closed balls containing K do not contain additional points.

Let $p \notin K$ be arbitrary. Since K is compact, for a suitable $R > 0$ we have $K \subset B_R(p)$. On the other hand, as K is compact and convex, p can be strictly separated from K by a hyperplane. Let M denote the intersection of $B_R(p)$ and the closed half space, bounded by H and containing K. Clearly, $K \subseteq M$, implying that if we find a closed ball containing M and not containing p, we have shown the statement. Let L denote the half line starting at p, perpendicular to H and intersecting H. Let h denote the distance of p and H. Then M is a section of a ball smaller than a half ball, and the radius of the base of M is $\sqrt{R^2 - h^2}$ by the Pythagorean Theorem. Let x denote the point in L at the distance from H equal to h_x which is separated from p by H. Observe that by our previous consideration $M \subseteq B_r(x)$ if and only if $M \cap H \subseteq B_r(x)$, that is, if $r \geq \sqrt{R^2 - h^2 + h_x^2}$. On the other hand, $p \notin B_r(x)$ if and only if $r < h_x + h$. But $(h_x+h)^2 - (R^2 - h^2 + h_x^2) = 2hh_x + 2h^2 - R^2 > 0$ if h_x is sufficiently small, and thus, there is some value of r satisfying both inequalities. This implies the statement.

Exercise 2. Let $K \subset \mathbb{R}^n$ be a compact set. We have shown that if K is convex, then it is supported at every boundary point by a hyperplane. Can this statement be reversed; e.g. if K is supported at every boundary point by a hyperplane, then K is convex?

Solution

If $\dim(K) = \dim \operatorname{aff}(K) < n$, then there is a hyperplane H with $K \subset H$. But then H supports K at every boundary point, and thus, the condition is satisfied, even if K is not convex. Thus, the answer is negative. There are other examples as well, e.g. taking the boundary of an arbitrary, compact, convex set with nonempty interior (e.g. a closed ball), or the set of vertices of a polytope (e.g. a cube).

Exercise 3. Let $K \subset \mathbb{R}^n$ be a compact, convex set, and let F be a face of K. Prove that if p is an extremal point of F, then p is an extremal point of K.

Solution

If $p \in F$, then $F \neq \emptyset$ K is a proper face. But then by the definition of face, there is a linear functional $f : \mathbb{R}^n \to \mathbb{R}$ that attains its minimum on K exactly at F. But then the statement is the consequence of Theorem 1 of fifth lecture.

Exercise 4. Let K be a compact, convex set, and let $p \in K$ be a point for which $||p|| \geq ||q||$ for any $q \in K$. Prove that then $p \in \operatorname{ex} K$.

Solution

Let $r = ||p||$. By the conditions, $K \subseteq B_r(o)$, where $B_r(o)$ denotes the closed ball of radius r and center o, and $p \in \operatorname{bd} B_r(o)$, and thus, $B_r(o)$ has a supporting (tangent) hyperplane H which intersects the ball at the point p. The only common point of $B_r(o)$ and H is p, but then $p \in K \subseteq B_r(o)$ yields $K \cap H = \{p\}$, and thus, $p \in \operatorname{ex}(K)$.

Exercise 5. Let $A \subset \mathbb{R}^n$ be compact. Verify that $p \in A$ is an extremal point of $\operatorname{conv}(A)$ if and only if $p \notin \operatorname{conv}(A \setminus \{p\})$.

Solution

Assume that $p \in \operatorname{ext}\operatorname{conv}(A \setminus \{p\})$. Then p can be written as a convex combination of points from $A \setminus \{p\}$; that is, there are points $p_1, p_2, \dots, p_k \in A \setminus \{p\}$ and coefficients $0 \leq \lambda_1, \dots, \lambda_k$, $\sum_{i=1}^k \lambda_i = 1$ such that $p = \sum_{i=1}^k \lambda_i p_i$. We can assume that we choose a combination in which k is minimal. In this case every coefficient is from $(0,1)$, and since $p \notin A \setminus \{p\}$, we have $k \geq 2$. Let $q = \frac{1}{1-\lambda_1} \sum_{i=2}^k \lambda_i p_i$. According to our assumption for k, $q \neq p$, but

$$p = \sum_{i=1}^k \lambda_i p_i = \lambda_1 p_1 + (1-\lambda_1) q.$$

Since $q \in \operatorname{conv}(A \setminus \{p\}) \subseteq \operatorname{conv} A$, we managed to express p as a relative interior point of a segment in $\operatorname{conv}(A)$, yielding that $p \notin \operatorname{ext}(\operatorname{conv}(A))$.

Now we will show that if $p \notin \operatorname{ext}(\operatorname{conv}(A))$, then $p \in \operatorname{conv}(A \setminus \{p\})$. We do it by induction on n. If $n = 1$, then $\operatorname{conv}(A)$ is a closed segment, whose extremal points are its endpoints, which readily implies the statement.Assume that the statement holds for any compact set in an $(n-1)$-dimensional Euclidean space. Let $A \subset \mathbb{R}^n$, and assume that $p \notin \operatorname{ext}(\operatorname{conv}(A))$. If $p \in \operatorname{bd}\operatorname{conv}(A)$, then $\operatorname{conv}(A)$ has a supporting hyperplane H containing p. But, by Proposition 1 of the second lecture, $\operatorname{conv}(A \setminus \{p\}) \cap H = \operatorname{conv}(H \cap (A \setminus \{p\}))$, and by Theorem 1 of the fifth lecture, $H \cap \operatorname{ext}(\operatorname{conv}(A)) = \operatorname{ext}(H \cap \operatorname{conv}(A))$. Thus $p \notin \operatorname{ext}(\operatorname{conv}(H \cap A))$, which, by the induction hypothesis, implies that $p \in \operatorname{conv}((A \cap H) \setminus \{p\}) \subseteq \operatorname{conv}(A \setminus \{p\})$. From this the statement follows. Assume now that $p \in \operatorname{int}\operatorname{conv}(A)$. Then there are $q, r \in \operatorname{conv}(A)$ $p \neq q, r$ such that $p \in [q, r]$, where, withput loss of generality, we may assume that $q, r \in \operatorname{bd}\operatorname{conv}(A)$. Let H_q and H_r be supporting hyperplanes of $\operatorname{conv}(A)$ satisfying $q \in H_q$ and $r \in H_r$. But $p \in \operatorname{int}\operatorname{conv}(A)$ implies $p \notin H_q$ and $p \notin H_r$. Hence, $q \in H_q \cap \operatorname{conv}(A) = \operatorname{conv}(H_q \cap A) \subseteq \operatorname{conv}(A \setminus \{p\})$, and it follows similarly that $r \in \operatorname{conv}(A \setminus \{p\})$, which implies the statement.

Exercise 6. Prove that every exposed point is also an extremal point.

Solution

This is exactly Proposition 2 of the fifth lecture.