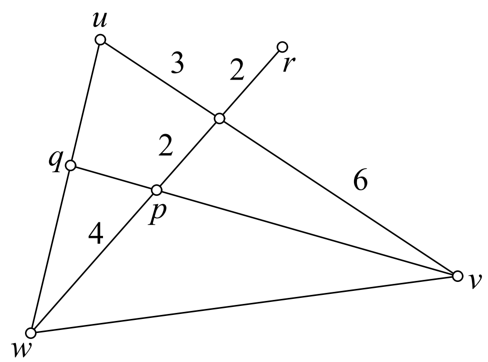
Convex Geometry tutorial
For students with mathematics major

Problem sheet 1 - Affine and convex combinations

Exercise 1. In the picture numbers denote lengths of segments. Express the points p, q, r as affine combinations of the points u, v, w.



Exercise 2. Let F_1 and F_2 be affine subspaces of $\mathbb{R}^n$. Assume that $F_1 \cap F_2 \neq \emptyset$. Prove that then $\dim F_1 + \dim F_2 - n \leq \dim(F_1 \cap F_2)$.

Exercise 3. Let $K \subseteq \mathbb{R}^n$ be a convex set. Prove that $\operatorname{int} K$ and $\operatorname{cl} K$ are convex sets.

Exercise 4. Verify that the intersection of arbitrarily many convex sets is convex.

Exercise 5. Let $x_1, \ldots, x_k \in \mathbb{R}^n$, és $\alpha_1, \alpha_2, \ldots, \alpha_k \in \mathbb{R}$. Prove that the set

$$P = \{y \in \mathbb{R}^n : \langle y, x_i \rangle \leq \alpha_i, i = 1, 2, 3 \ldots, k\}$$

is convex. Is it true in case of infinitely many inequalities?

Exercise 6. Let $S \subseteq \mathbb{R}^n$ be arbitrary. Let the *kernel* of S be the set of points x with the property that $[x, y] \subseteq S$ holds for any $y \in S$. Prove that the kernel of S is convex.

Exercise 7. Let $K \subseteq \mathbb{R}^n$ be convex, and let $T : \mathbb{R}^n \to \mathbb{R}^n$ be an invertible linear transformation. Prove that the set $T(K) = \{T(x) : x \in K\}$ is convex. Prove that for the set P defined in Exercise 5 there are vectors $w_1, w_2, \ldots, w_k \in \mathbb{R}^n$ and numbers $\beta_1, \beta_2, \ldots, \beta_k \in \mathbb{R}$ such that

$$T(P) = \{y \in \mathbb{R}^n : \langle y, w_i \rangle \leq \beta_k, i = 1, 2, 2 \ldots, k\}.$$