Convex Geometry tutorial
For students with mathematics major

Problem sheet 2 - Convex hull, theorems of Radon and Carathéodory

Exercise 1. Prove that if $A \subseteq B$, then $\operatorname{conv} A \subseteq \operatorname{conv} B$.

Exercise 2. A set $S \subseteq \mathbb{R}^n$ is called a *convex cone* if it is convex and for every $x \in S$ the points λx, $\lambda \geq 0$ are elements of S. Following the definition of convex combination and convex hull, define the conic combination of points and the conic hull of a set. Show that a conic hull is a convex cone, and it coincides with the set of the conic combinations of the finite subsets of the set.

Exercise 3. A set $K \subset \mathbb{R}^n$ is called *locally convex* if for every $p \in K$ there is some $\rho > 0$ such that the intersection of K with the ball $B(p, \rho)$ of radius ρ and center p is convex. Is it true that every locally convex set is convex?

Exercise 4. Give an example for a closed set $A \subseteq \mathbb{R}^2$ whose convex hull is not closed.

Exercise 5. Prove that the convex hull of an open set is open.

Exercise 6. Let $S \subset \mathbb{R}^n$ be a set consisting of $n + 2$ points in general position (i.e. any $n + 1$ of the points is affinely independent). Prove that then S can be uniquely decomposed into two disjoint subsets S_1, S_2 satisfying $\operatorname{conv} S_1 \cap \operatorname{conv} S_2 \neq \emptyset$. In addition, prove that in this case the intersection is a singleton.

Exercise 7. * Let $\sigma \in S_n$ be a permutation. Define the permutation matrix assigned to σ by $A_\sigma := (a_{ij})$, where

$$a_{ij} = \begin{cases} 1, & \text{if } \sigma(i) = j \\ 0, & \text{if } \sigma(i) \neq j. \end{cases}$$

A matrix $B = (b_{ij})$ is called *doubly stochastic*, if its entries are nonnegative, and the sum of the entries in each row and each column is one. Prove that the convex hull of the set of permutation matrices in $\mathbb{R}^{n^2}$ is the set of doubly stochastic matrices. (Hint: try to reduce the problem to Hall's theorem for bipartite graphs)