Convex Geometry tutorial
For students with mathematics major

Problem sheet 3 - Helly's theorem

Exercise 1. Show that the finite version of Helly's theorem does not hold for nonconvex sets.

Exercise 2. Give an example of a family of infinitely many closed, convex sets in $\mathbb{R}^n$ with the property that any $n+1$ elements of the set intersect, but the intersection of all elements is empty.

Exercise 3. Give an example of a family of finitely many convex sets in the plane such that no two of them are disjoint, but the intersection of all of them is empty.

Exercise 4. Let $\mathcal{F}$ be a finite family of convex sets in $\mathbb{R}^n$, and let L be an affine subspace in $\mathbb{R}^n$. Prove that if for any at most $n+1$ elements of $\mathcal{F}$ a translate of L intersects all elements, then there is a translate of L that intersects all elements of $\mathcal{F}$.

Exercise 5. Let $\mathcal{F}$ be a finite family of convex sets and let C be a convex set in $\mathbb{R}^n$. Prove that if, for any at most $(n+1)$ elements of $\mathcal{F}$, there is a translate of C that intersects/contains/is contained in all of them, then then there is a translate of C that intersects/contains/is contained in all elements of $\mathcal{F}$.

Exercise 6. *(Krasnosselsky's art gallery theorem) Let $S \subset \mathbb{R}^n$ be a compact set of at least $n+1$ points, and assume that for any $p_1, p_2, \ldots, p_{n+1}$ there is some $q \in S$ from which every p_i is visible, or in other words, $[p_1, q], \ldots, [p_{n+1}, q] \subseteq S$. Prove that then S is *starlike*, that is, it contains a point from which every point of S is visible.