Convex Geometry tutorial
for students with mathematics major

Problem sheet 4 - Hyperplanes, Minkowski sum, separation

Exercise 1. Prove that if A and B are two disjoint, convex sets in $\mathbb{R}^n$, then there are disjoint convex sets A', B' in $\mathbb{R}^n$ satisfying $A \subset A'$, $B \subset B'$, and $A' \cup B' = \mathbb{R}^n$.

Exercise 2. Describe all decompositions of the 3-dimensional Euclidean space into the union of two disjoint, convex sets. What is the situation in $\mathbb{R}^n$?

Exercise 3. (a) Let T be a regular triangle. What is $T - T$? What is $T + T$?

(b) For any compact set $T \subset \mathbb{R}^n$ and positive integer k let $T_k = \frac{\overbrace{T + T + \ldots + T}^{k}}{k}$. Prove that if T is convex and $k \in \mathbb{Z}^+$, then

$$T_k = T.$$

If T is not necessarily convex, what is the relationship between T, T_k and $\mathrm{conv}(T)$?
(c)* Prove that if $T \subset \mathbb{R}^n$ is compact and convex and$k \in \mathbb{Z}^+$, then

$$V(T_k) \leq V(T_{k+1}),$$

where $V(\cdot)$ denotes n-dimensional volume (Lebesgue measure). What happens if T is not necessarily convex?

Exercise 4. Let the sum of the planar vectors $a_1, a_2, \ldots, a_k$ be o. Assume that among these vectors there are no two with the same direction. Prove that up to translation there is a unique convex polygon whose sides, oriented according to a fixed orientation of the plane, are exactly these vectors.

Exercise 5. Let K and L be convex polygons, whose edge vectors, according to a fixed orientation of the plane, are $a_1, \ldots, a_k$ and $b_1, \ldots, b_m$, respectively. Prove that if, among the vectors, there are no two in the same direction, then $K + L$ is a convex polygon whose edge vectors are exactly $a_1, \ldots, a_k, b_1, \ldots, b_m$. How can we modify the statement if there are vectors with the same direction?

Exercise 6. Let K and L be plane convex bodies. Prove that then $\mathrm{perim}(K + L) = \mathrm{perim}(K) + \mathrm{perim}(L)$.