Convex Geometry tutorial
for students with mathematics major

Problem sheet 6 - Algebra of convex sets, the Euler characteristic

Exercise 1. Prove that if $f, g \in \mathcal{K}(\mathbb{R}^n)$, then $fg \in \mathcal{K}(\mathbb{R}^n)$, and if $f, g \in \mathcal{C}(\mathbb{R}^n)$, then $fg \in \mathcal{C}(\mathbb{R}^n)$.

Exercise 2. Is it true that the indicator functions of compact, convex sets form a basis of the vector space $\mathcal{K}(\mathbb{R}^n)$?

Exercise 3. Let $K_1, K_2, K_3 \subset \mathbb{R}^n$ be closed, convex sets whose union is convex. Prove that if the intersection of any pair of them is nonempty, then the intersection of all three sets is nonempty.

Exercise 4. Let $K_1, K_2, \ldots K_m \subset \mathbb{R}^n$ be closed, convex sets whose union is convex. Prove that if any k of them have a nonempty intersection, then there are $k+1$ sets among them whose intersection is not empty.

Exercise 5. Define the volume $V(\cdot)$ of a set in the usual way, i.e. as its Lebesgue measure. Using the fact that (being Borel sets), any compact, convex set is Lebesgue measurable, prove that for any $\alpha_1, \alpha_2, \ldots, \alpha_m \in \mathbb{R}$ and compact, convex sets $K_1, K_2, \ldots, K_m \in \mathbb{R}^n$, if $\sum_{i=1}^m \alpha_i I[K_i] = 0$, then $\sum_{i=1}^m \alpha_i V(K_i) = 0$.

Exercise 6. Prove that there is a valuation on $\mathcal{K}(\mathbb{R}^n)$ whose value at the indicator function of any compact, convex set K is $V(K)$.