Convex Geometry tutorial for students with mathematics major

Problem sheet 7 - Polytopes, polyhedral sets, their face structures

Exercise 1. Prove that the extremal points of a polytope are also its exposed points.

Exercise 2. A proper face of a closed, convex set is its intersection with one of its supporting hyperplanes. Prove that a proper face of a polyhedral set is also a polyhedral set.

Exercise 3. Let $K = \{p : \langle p, u_i \rangle \geq \alpha_i, i \in I\}$, with $|I| < \infty$, be a bounded polyhedral set. For any point $p \in K$ let $I(p) = \{i : \langle p, u_i \rangle = \alpha_i\}$. Let $F = \{q \in K : \langle q, u_i \rangle = \alpha_i : i \in I(p)\}$. Prove that if we regard K as a face of itself, then F is the smallest face, with respect to inclusion, such that $p \in F$.

Exercise 4. Prove that every n-dimensional polytope has a facet. Prove that for every $k = 0, 1, \ldots, n-1$, every n-dimensional polytope has a k-dimensional face.

Exercise 5. Prove that an $(n-2)$-dimensional face of an n-dimensional polytope belongs to exactly two facets.

Exercise 6. (Diamond property) Let P be an arbitrary n-dimensional polytope, and $F \subset G$ are faces of P with $\dim F + 2 = \dim G$. Then P has exactly two faces F_1, F_2, different from F and G, that satisfy $F \subset F_1, F_2 \subset G$.