Convex Geometry tutorial for students with mathematics major

Problem sheet 8 - Euler's theorem

Exercise 1. Let $P \subset \mathbb{R}^n$ be an n-dimensional convex polytope. Let H be a hyperplane, passing through an interior point of P, which does not contain any vertex of P. Let H^+ be one of the two open half spaces bounded by H, and let f_i^+ denote the number of the i-dimensional faces of P contained in H^+. Then

$$\sum_{i=0}^{n-1}(-1)^i f_i^+ = 1.$$

Exercise 2. Let $P \subset \mathbb{R}^n$ be an n-dimensional convex polytope, and let $f : \mathbb{R}^n \to \mathbb{R}$ be a linear functional with mutually different values at the vertices of P. For any vertex x let f_i^x P denote the number of the i-dimensional faces F of P that satisfy $f(x) = \max\{f(y) : y \in F\}$. Prove that

$$\sum_{i=0}^{n-1}(-1)^i f_i^x = \begin{cases} 1 & \text{if } f(x) \text{ is the minimum of } f \text{ on } P, \\ (-1)^{n-1} & \text{if } f(x) \text{ is the maximum of } f \text{ on } P, \\ 0 & \text{otherwise .} \end{cases}$$

Exercise 3. Let $P \subset \mathbb{R}^n$ be an n-dimensional convex polytope, and let F be a k-dimensional face of P. Let $f_j(F, P)$ denote the number of the j-dimensional faces of P containing F. Prove that

$$\sum_{j=k}^{n-1}(-1)^j f_j(F, P) = (-1)^{n-1}.$$

Exercise 4. The f-vector of an n-dimensional convex polytope is $(f_0, f_1, \ldots, f_{n-1}, 1) \in \mathbb{R}^{n+1}$, where f_i denotes the number of the i-dimensional faces of the polytope. Show that the affine hull of the set of the f-vectors of all 3-dimensional polytopes is a plane, or in other words, apart from Euler's formula, there is no other nontrivial linear dependence relation between the face numbers holding for every 3-dimensional polytope.