Final exam topics

1. Characteristic of a ring, Frobenius endomorphism, existence and uniqueness of the prime field, degree of a field extension, finite extensions, multiplicativity formula, algebraic and transcendent elements, minimal polynomial, algebraic field extensions, characterizations of finite and algebraic extensions
2. Algebraic closure, extensions of field homomorphisms, uniqueness of algebraic closure, splitting fields: existence and uniqueness, normal extensions, equivalent characterizations, normal closure: existence and uniqueness
3. Separable polynomials, separable extensions, irreduciblilty and separability, perfect fields, separability degree, multiplicativity formula for the separability degree, equivalent characterizations of separable extensions, separable extensions are simple
4. Existence and uniqueness of finite fields, finite fields as splitting fields, finite fields are perfect, $\mathbb{F}_q$-automorphisms of an extension of finite fields, number of irreducible polynomials over finite fields
5. Galois extension and Galois group, fundamental theorem of Galois theory, fundamental theorem of algebra (with the sketch of the proof focusing on the application of the fundamental theorem), Galois groups of polynomials as subgroups of symmetric groups, Galois group of a finite Galois extension is isomorphic to a subgroup of a symmetric group
6. Group of roots of unities, primitive nth roots, cyclotomic fields, their Galois groups in general and over $\mathbb{Q}$, cyclotomic polynomials and their basic properties, cyclotomic polynomials over finite fields, an application: a special case of Dirichlet's theorem on primes in arithmetic progressions and its application to inverse Galois theory (and the role of Dirichlet's theorem in the proof)
7. Definition of norm and trace, expression of their value via field homomorphisms, linear independence of characters, non-triviality of the trace (with proof), Hilbert's Theorem 90, multiplicative and additive forms, characterization of cyclic extensions, Artin–Schreier theorem
8. Applications of Galois theory: characterizations of constructable numbers, squaring the circle, doubling the cube, construction of regular n-gons (and the role of cyclotomic fields in the proof), solution of the cubic equation, solvability by radicals and solvability, the equivalence of the two notions, the general polynomial equation of degree n is not solvable if $n \geq 5$
9. Modules: definition, basic examples, module homomorphisms, submodule, fundamental theorem on homomorphisms, basic constructions, linear independence, basis, free module, torsion elements, torsion submodule, rank, existence and uniqueness of elementary divisors for a submodule of a free module fundamental theorem of finitely generated modules over principal ideal domains, fundamental theorem of finitely generated abelian groups