Algebraic and Analytic Number Theory

1. **Primes, the prime number theorem.** State the prime number theorem (with error term as well). The Riemmann zeta function, the functional equation, the roots of the Riemann zeta function, the Riemann hypothesis. Asymptotics for the n-th prime. Twin primes, twin prime conjecture. Upper estimation for the number of twin primes. Sum of reciprocals of the primes and the twin primes.

2. **Primes in arithmettic progressions.** State the Dirichlet's theorem. The definitions of the Dirichlet series and the Dirichlet L function. The product form of the Dirichlet's L function. The Generalized Riemann Hypothesis. The definition of the character and the Dirichlet character.

3. **Exponential sums.** Definition of the exponential sums and the Gaussian sums. Estimation of the Gaussian sums. The number of solutions of the congruence $f(x_1, \ldots, x_r) \equiv 0 \bmod m$. Lagrange's theorem about quadratic congruences, the quadratic reciprocity law.

4. **The circle method. The circle problem.** The outline of the circle method. The Waring problem. The binary and ternary Goldbach conjecture. The circle problem. The simple asymptotics for the circle problem. The theorem of Erdős and Fuchs and application for the circle problem.

5. **The *abc* conjecture and its corollaries. The quadratic reciprocity law.** State the *abc* conjecture. The applications of the *abc* conjecture: the generalized Fermat equation, Erdős's theorem about 4-powerful numbers.

6. **The quadratic extensions.** Ring of algebraic integers. The quadratic fields. Description of the rings of algebraic integers in the quadratic fields. Description of units of quadratic fields. Definitions of norm, trace and discriminant. The Dirichlet unit theorem.

7. **Ideals. The ideal class group.** Definition of the norm of an ideal, and its properties. The product of ideals. Norm of a principal ideal. Unique factorization of ideals. Definition of the fractional ideal. The ideal class group and its properties, the class number. The definition of the regular prime. Fermat's last theorem for regular primes.

8. **The Ramification Theory.** Definition of ramified, unramified and inert primes with examples. Definition of the ramification index. The ramified primes and the discriminants.