Differential topology, 2024/25 spring

1. The tangent bundle of a smooth manifold and its sections (definition, interpretation and examples); pushforward of a vector field [5, §1] [2, Section 4.2].

2. Vector bundles and their sections (definition and examples); constructing new vector bundles out of old (extension of linear algebraic operations over vector bundles) [5, §2, §3].

3. Elements of linear algebra (tensor and exterior powers of vector spaces; general tensors and alternating forms; wedge product; Grassmann algebra; Hodge operation) [4, Chapters XIII, XVI,XIX] [2, Section 1.2].

4. Differential forms (their local and global shapes, examples); pullback of differential forms; existence and uniqueness of exterior differentiation on a manifold [1, §1] [2, Subsections 4.6.1-4.6.2].

5. Oriented manifolds and manifolds-with-boundary; integration on manifolds; Stokes' theorem [1, §3] [2, Section 4.7].

6. de Rham cohomology groups of a manifold (definition, interpretations); Poincaré's lemma [1, §1, §4] [2, Subsection 4.6.3].

7. Elements of homological algebra (exact sequences, snake lemma: the long exact sequence on cohomology induced by a short exact sequence); Mayer–Vietoris exact sequence of de Rham cohomology groups (examples) [1, §2].

8. Notion of an oriented Riemannian manifold; exterior codifferentiation, definition of the Laplace and the Dirac–Euler operator on manifolds; definition of harmonic k-forms (examples) [7, Vol I Chapter 2 §10].

9. Various norms ($\|\cdot\|_{C^k}$ and $\|\cdot\|_{L^2_s}$) and Sobolev space structures on the space of sections of vector bundles, the dual Sobolev spaces (distributions on manifolds); a special case of the Sobolev embedding theorem: the embedding $L^2_s(M;E) \subset C^k(M;E)$ is continuous (if $s > \frac{1}{2}\dim M + k$), corollary: $\bigcap\limits_s L^2_s(M;E) = C^\infty(M;E)$ [8, Chapter IV §1].

10. Definition of a compact self-adjoint operator; the spectral theorem for compact self-adjoint operators on a separable Hilbert space [7, Vol I Appendix A §6].

11. A special case of the Rellich–Kondrashov compactness theorem: the embedding $L^2_s(M;E) \subset L^2_t(M;E)$ is compact (if M is compact and $s > t$, $s \geq 0$) [8, Chapter IV §1].

12. Elliptic regularity for the Laplace and the Dirac–Euler operator, consequences: smoothness of eigen-forms, Vitali–Montel-type theorems [7, Vol I Chapter 5 §1].

13. Hodge decomposition theorem and its consequences: finite dimensionality of the de Rham cohomology groups (as vector spaces), Poincaré duality, Künneth formula [7, Vol I Chapter 5 §8].

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